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Finite Groups with Sylow 2-Subgroups of Type \mathcal{O}_{12}

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Let \mathcal{O}_n denote the alternating group on n letters.

A 2-group is said to be of *type X* if it is isomorphic to a Sylow 2-subgroup of the group X . In this paper we study finite groups with Sylow 2-subgroups of type \mathcal{O}_{12} .

If G is a group with Sylow 2-subgroup, S , of type X , we say that G has the *involution fusion pattern* of X if for some isomorphism θ of S onto a Sylow 2-subgroup of X , two involutions a, b of S are conjugate in G if and only if the involutions $a\theta, b\theta$ of $S\theta$ are conjugate in X .

Finally we say that a group G is *fusion-simple* if $G = O^2(G)$ and $O_2(G) = Z(G) = \langle 1 \rangle$.

The object of this paper is to prove the following:

MAIN THEOREM. *Let G be a fusion-simple finite group with Sylow 2-subgroups of type \mathcal{O}_{12} . Then one of the following holds:*

- (i) $G \cong \mathcal{O}_{12}$ or \mathcal{O}_{13} ,
- (ii) $G \cong \text{Sp}_6(2)$,
- (iii) $G \cong L_2(7) \cdot E_{64}$, the normalizer of an elementary group of order 2^6 in $\text{Sp}_6(2)$,
- (iv) $G \cong \mathcal{O}_7 \cdot E_{64}$, the split extension of an elementary group of order 2^6 by an \mathcal{O}_7 with the action afforded by the 6-dimensional irreducible $GF(2)$ -representation,
- (v) G has the involution fusion pattern of $\Omega_7(3)$.

Here, $\text{Sp}_6(2)$ denotes the group of all 6×6 symplectic matrices over $GF(2)$ and $\Omega_7(3)$ denotes the commutator subgroup of the group of all 7×7 orthogonal matrices over $GF(3)$.

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In the process of proving the Main Theorem, we obtain the following characterizations:

THEOREM A. *If G is a finite group with the involution fusion pattern of $\mathrm{Sp}_6(2)$, then $G/O(G) \cong \mathrm{Sp}_6(2)$.*

THEOREM B. *If G is a finite group with the involution fusion pattern of \mathcal{O}_{12} , then $G/O(G) \cong \mathcal{O}_{12}$ or \mathcal{O}_{13} .*

For most of the notation and the fundamental background knowledge the reader is referred to [10]. We denote by \mathfrak{S}_n the symmetric group of degree n ; by $\mathrm{Syl}_p(G)$ the collection of all Sylow p -subgroups of G ; by $\mathcal{U}_G(A, 2')$ the collection of all odd-order subgroups of G normalized by A ; by $R[G]$ the group ring of G over R ; by E_n the elementary abelian group of order n ; and by $U_3(2^n)$ the projective special group of 3×3 unitary matrices over $GF(2^{2n})$. $U \mid V$ signifies that U is isomorphic to a component of V .

1. PRELIMINARIES

A Sylow 2-subgroup \tilde{S} of \mathcal{O}_{12} is generated by the involutions

$$\begin{aligned}\tilde{\pi}_1 &= (1\ 2)(3\ 4), & \tilde{\pi}_1' &= (1\ 3)(2\ 4), \\ \tilde{\pi}_2 &= (5\ 6)(7\ 8), & \tilde{\pi}_2' &= (5\ 7)(6\ 8), \\ \tilde{\pi}_3 &= (9\ 10)(11\ 12), & \tilde{\pi}_3' &= (9\ 11)(10\ 12), \\ \tilde{\mu} &= (1\ 2)(5\ 6), & \tilde{\mu}' &= (1\ 2)(9\ 10), \\ \tilde{\tau} &= (1\ 5)(2\ 6)(3\ 7)(4\ 8).\end{aligned}$$

Let G be a group with Sylow 2-subgroup S of type \mathcal{O}_{12} . In the isomorphism from \tilde{S} onto S let the images of $\tilde{\pi}_i$, $\tilde{\pi}_i'$, $\tilde{\mu}$, $\tilde{\mu}'$, $\tilde{\tau}$ be π_i , π_i' , μ , μ' , τ , respectively. In Section 2, the structure of S is examined.

Goldschmidt's theorem [7] on conjugation families is applied in Section 3 to prove

THEOREM 3.3. *Suppose that B_0 and B_1 are subsets of S which are conjugate in G . Then there exist elements x_1, x_2, \dots, x_n with either $x_i \in N_G(J(S))$ or $x_i \in C_G(Z(S))$ such that*

- (a) $B_0^{x_1 x_2 \cdots x_i} \subseteq S$ for $1 \leq i \leq n$,
- (b) $B_0^{x_1 x_2 \cdots x_n} = B_1$.

The structure of $N_G(J(S))/C_G(J(S))$ is determined in Section 4. Further analysis is divided into two cases according as $C_G(Z(S))/O(C_G(Z(S)))$ is or is not covered by $N_G(J(S))$.

A subgroup M of a group G is said to be *weakly embedded* in G if M contains a Sylow 2-subgroup, S , of G such that

- (i) $N_G(S) = O(N_G(S)) \cdot (M \cap N_G(S))$,
- (ii) $C_G(t) = O(C_G(t)) \cdot (M \cap C_G(t))$ for every involution, t , of S .

Goldschmidt has proven the following theorem [9]:

Suppose A is a 2-subgroup of the finite group G and $N_G(A)$ is weakly embedded in G . Then either $G = N_G(A) \cdot O(G)$ or $G/O(G) \cong \text{Aut } G_1$, where G_1 is a normal subgroup of $G/O(G)$ containing $AO(G)/O(G)$ and G_1 is isomorphic to one of the following groups:

- (a) $L_2(q)$, $q \equiv 0, 3, 5 \pmod{8}$,
- (b) $Sz(2^{2n+1})$, $n \geq 1$,
- (c) $U_3(2^n)$, $n \geq 2$.

In the case where $C_G(Z(S))/O(C_G(Z(S)))$ is covered by $N_G(J(S))$, it is proved in Section 5 that $N_G(J(S))$ is weakly embedded in G . Then by Goldschmidt's theorem and the known fusion of involutions in G , $G = N_G(J(S))$ in this case.

The remaining case is settled in Section 6. Walter's classification of groups with abelian Sylow 2-subgroups and characterization theorems of Gorenstein and Harada permit the determination of $C_G(t)/O(C_G(t))$ for all involutions t in S . As Yamaki [17, 18] has characterized \mathcal{O}_{12} and \mathcal{O}_{13} by the centralizer of $(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)$ and the involution fusion pattern and $\text{Sp}_6(2)$ by the centralizer of

$$\begin{pmatrix} 1 & & & & & & 1 \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix}$$

and the involution fusion pattern, it remains to prove that

$$O(C_G(\pi_1\pi_2\pi_3)) = \langle 1 \rangle.$$

This is accomplished by a signalizer functor argument.

We say that θ is an A -*signalizer functor* on G if A is an elementary abelian 2-subgroup of G and if to each involution, x , of A there is associated a subgroup $\theta(C_G(x))$ of $O(C_G(x))$ such that

- (a) $\theta(C_G(x))$ is normalized by A .
- (b) $\theta(C_G(x)) \cap C_G(y) \subseteq \theta(C_G(y))$ for all $x, y \in A^\#$.

A theorem of Goldschmidt [8] states

If G possesses the A -signalizer functor θ and A has rank at least 3, then the subgroup $W_A = \langle \theta(C_G(x)) \mid x \in A^\# \rangle$ of G has odd order.

Now when S is a rank 2-group of type \mathcal{O}_{12} , $J(S)$ is elementary abelian of rank 6. In the cases under consideration it is proved that O is a $J(S)$ -signalizer functor. It follows that $N_G(W_{J(S)})$ is strongly embedded in G , if $W_{J(S)} \neq \langle 1 \rangle$. Since a group possessing a strongly embedded subgroup has only one class of involutions, the known fusion of involutions in G forces $W_{J(S)} = \langle 1 \rangle$. Thus $O(C_G(\pi_1\pi_2\pi_3)) = \langle 1 \rangle$, completing the proof.

2. ON 2-GROUPS OF TYPE \mathcal{O}_{12}

Let G be a finite group with Sylow 2-subgroups of type \mathcal{O}_{12} .

Let $S \in \text{Syl}_2(G)$.

S is generated by the involutions $\pi_1, \pi_2, \pi_3, \pi_1', \pi_2', \pi_3', \mu, \mu', \tau$ subject to the relations

$$\begin{aligned} [\mu, \pi_1'] &= [\mu', \pi_1'] = \pi_1; \\ [\mu, \pi_2'] &= \pi_2; \\ [\mu', \pi_3'] &= \pi_3; \\ [\tau, \pi_1] &= [\tau, \pi_2] = \pi_1\pi_2; \\ [\tau, \pi_1'] &= [\tau, \pi_2'] = \pi_1'\pi_2'; \\ [\tau, \mu'] &= \mu; \end{aligned}$$

all other commutators of the generating involutions trivial.

Let $A = \langle \pi_1, \pi_2, \pi_3, \pi_1', \pi_2', \pi_3' \rangle$.

Then A is the unique elementary abelian subgroup of S of rank at least 6. Thus, in particular, A is weakly closed in S with respect to G , whence $N_G(A)$ controls fusion of elements of A in G .

S is a split extension of A by the dihedral group, $D = \langle \mu, \mu', \tau \rangle$, where $\langle \mu \rangle = Z(D)$.

The lower central series of S is

$$\begin{aligned} S \supseteq S^1 &= \langle \pi_1, \pi_2, \pi_3, \pi_1'\pi_2', \mu \rangle \supseteq S^2 = \langle \pi_1, \pi_2 \rangle \supseteq S^3 \\ &= \langle \pi_1\pi_2 \rangle \supseteq \langle 1 \rangle. \end{aligned}$$

The upper central series of S is

$$\begin{aligned} \langle 1 \rangle \trianglelefteq Z(S) &= \langle \pi_1\pi_2, \pi_3 \rangle \trianglelefteq Z_2(S) = \langle \pi_1, \pi_2, \pi_3, \pi_3' \rangle \trianglelefteq Z_3(S) \\ &= \langle \pi_1, \pi_2, \pi_3, \pi_1'\pi_2', \pi_3', \mu \rangle \trianglelefteq S. \end{aligned}$$

LEMMA 2.1. $N_G(S) = SC_G(S)$.

Proof. Consider the normal series for S :

$$S \supseteq A \supseteq Z_3(S) \cap A \supseteq Z_2(S) \supseteq Z_2(S) \cap S^1 \supseteq Z(S) \supseteq S^3 \supseteq \langle 1 \rangle.$$

As every factor group is isomorphic to either D_8 nor Z_2 and as neither D_8 nor Z_2 admits a nontrivial automorphism of odd order, every element of $N_G(S)$ of odd order stabilizes this normal series.

The lemma follows.

COROLLARY 2.2. $\pi_3 \not\sim_G \pi_1\pi_2 \not\sim_G \pi_1\pi_2\pi_3 \not\sim_G \pi_3$.

Proof. By Burnside's lemma, $N_G(S)$ controls fusion in $Z(S)$.

Notation. Set

$$\alpha = \pi_1\pi_2\pi_3, \quad \alpha' = \pi_1'\pi_2'\pi_3'.$$

The Conjugacy Classes of Involutions in S

- (1) $\pi_1 \sim \pi_2$
- (2) π_3
- (3) $\pi_1' \sim \pi_1\pi_1' \sim \pi_2' \sim \pi_2\pi_2'$
- (4) $\pi_3' \sim \pi_3\pi_3'$
- (5) $\pi_1\pi_2$
- (6) $\pi_1\pi_3 \sim \pi_2\pi_3$
- (7) $\pi_1\pi_2' \sim \pi_2\pi_1' \sim \pi_1\pi_2\pi_1' \sim \pi_1\pi_2\pi_2'$
- (8) $\pi_1\pi_3' \sim \pi_1\pi_3\pi_3' \sim \pi_2\pi_3' \sim \pi_2\pi_3\pi_3'$
- (9) $\pi_3\pi_1' \sim \pi_3\pi_2' \sim \pi_1\pi_3\pi_1' \sim \pi_2\pi_3\pi_2'$
- (10) $\pi_1'\pi_2' \sim \pi_1\pi_1'\pi_2' \sim \pi_2\pi_1'\pi_2' \sim \pi_1\pi_2\pi_1'\pi_2'$
- (11) $\pi_1'\pi_3' \sim \pi_2'\pi_3' \sim \pi_1\pi_1'\pi_3' \sim \pi_1\pi_3\pi_1'\pi_3' \sim \pi_2\pi_2'\pi_3' \sim \pi_2\pi_3\pi_2'\pi_3'$
 $\sim \pi_3\pi_2'\pi_3' \sim \pi_3\pi_1'\pi_3'$
- (12) $\pi_1\pi_2\pi_3$
- (13) $\pi_1\pi_2\pi_3' \sim \pi_1\pi_2\pi_3\pi_3'$
- (14) $\pi_1\pi_3\pi_2' \sim \pi_2\pi_3\pi_1' \sim \pi_1\pi_3\pi_3\pi_2' \sim \pi_1\pi_2\pi_3\pi_1'$
- (15) $\pi_1\pi_2'\pi_3' \sim \pi_2\pi_1'\pi_3' \sim \pi_1\pi_2\pi_2'\pi_3' \sim \pi_1\pi_3\pi_2'\pi_3' \sim \pi_1\pi_3\pi_1'\pi_3'$
 $\sim \pi_2\pi_3\pi_1'\pi_3' \sim \pi_1\pi_2\pi_3\pi_1'\pi_3' \sim \pi_1\pi_2\pi_3\pi_2'\pi_3'$
- (16) $\pi_3\pi_1'\pi_2' \sim \pi_1\pi_3\pi_1'\pi_2' \sim \pi_1\pi_2\pi_3\pi_1'\pi_2' \sim \pi_2\pi_3\pi_1'\pi_2'$
- (17) $\pi_1'\pi_2'\pi_3' \sim \pi_1\pi_3\pi_1'\pi_2'\pi_3' \sim \pi_2\pi_3\pi_1'\pi_2'\pi_3' \sim \pi_1\pi_2\pi_1'\pi_2'\pi_3'$
- (18) $\pi_1\pi_1'\pi_2'\pi_3' \sim \pi_2\pi_1'\pi_2'\pi_3' \sim \pi_3\pi_1'\pi_2'\pi_3' \sim \pi_1\pi_2\pi_3\pi_1'\pi_2'\pi_3'$
- (19) $\mu \sim \mu\pi_1 \sim \mu\pi_2 \sim \mu\pi_1\pi_2$
- (20) $\mu\pi_3 \sim \mu\pi_1\pi_3 \sim \mu\pi_2\pi_3 \sim \mu\pi_1\pi_2\pi_3$
- (21) $\mu\pi_3' \sim \mu\pi_1\pi_3' \sim \mu\pi_1\pi_3\pi_3' \sim \mu\pi_2\pi_3' \sim \mu\pi_2\pi_3\pi_3' \sim \mu\pi_3\pi_3'$
 $\sim \mu\pi_1\pi_2\pi_3' \sim \mu\pi_1\pi_2\pi_3\pi_3'$

- (22) $\mu' \sim \mu'\pi_1 \sim \mu'\pi_3 \sim \mu'\pi_1\pi_3 \sim \mu'\mu \sim \mu'\mu\pi_2 \sim \mu'\mu\pi_3 \sim \mu'\mu\pi_2\pi_3$
- (23) $\mu'\pi_2 \sim \mu'\pi_1\pi_2 \sim \mu'\pi_2\pi_3 \sim \mu'\pi_1\pi_2\pi_3 \sim \mu'\mu\pi_1 \sim \mu'\mu\pi_1\pi_2$
 $\sim \mu'\mu\pi_1\pi_3 \sim \mu'\mu\pi_1\pi_2\pi_3$
- (24) $\mu'\pi_2' \sim \mu'\pi_1\pi_2' \sim \mu'\pi_2\pi_2' \sim \mu'\pi_1\pi_2\pi_2' \sim \mu'\pi_3\pi_2' \sim \mu'\pi_1\pi_3\pi_2'$
 $\sim \mu'\pi_2\pi_3\pi_2' \sim \mu'\pi_1\pi_2\pi_3\pi_2' \sim \mu'\mu\pi_1' \sim \mu'\mu\pi_1\pi_1' \sim \mu'\mu\pi_2\pi_1'$
 $\sim \mu'\mu\pi_1\pi_2\pi_1' \sim \mu'\mu\pi_3\pi_1' \sim \mu'\mu\pi_1\pi_3\pi_1' \sim \mu'\mu\pi_2\pi_3\pi_1'$
 $\sim \mu'\mu\pi_1\pi_2\pi_3\pi_1'$
- (25) $\tau \sim \tau\pi_1\pi_2 \sim \tau\pi_1'\pi_2' \sim \tau\pi_1\pi_2\pi_1'\pi_2' \sim \tau\mu \sim \tau\mu\pi_1\pi_2 \sim \tau\mu\pi_1\pi_1'\pi_2'$
 $\sim \tau\mu\pi_2\pi_1'\pi_2'$
- (26) $\tau\pi_3 \sim \tau\pi_1\pi_2\pi_3 \sim \tau\pi_1'\pi_2'\pi_3 \sim \tau\pi_1\pi_2\pi_3\pi_1'\pi_2' \sim \tau\mu\pi_3 \sim \tau\mu\pi_1\pi_2\pi_3$
 $\sim \tau\mu\pi_1\pi_3\pi_1'\pi_2' \sim \tau\mu\pi_2\pi_3\pi_1'\pi_2'$
- (27) $\tau\pi_3' \sim \tau\pi_1\pi_2\pi_3' \sim \tau\pi_1'\pi_2'\pi_3' \sim \tau\pi_1\pi_2\pi_1'\pi_2'\pi_3' \sim \tau\mu\pi_3\pi_3'$
 $\sim \tau\mu\pi_1\pi_2\pi_3\pi_3' \sim \tau\mu\pi_1\pi_3\pi_1'\pi_2'\pi_3' \sim \tau\mu\pi_2\pi_3\pi_1'\pi_2'\pi_3'$
- (28) $\tau\pi_3\pi_3' \sim \tau\pi_1\pi_2\pi_3\pi_3' \sim \tau\pi_3\pi_1'\pi_2'\pi_3' \sim \tau\pi_1\pi_2\pi_3\pi_1'\pi_2'\pi_3'$
 $\sim \tau\mu\pi_3' \sim \tau\mu\pi_1\pi_2\pi_3' \sim \tau\mu\pi_1\pi_1'\pi_2'\pi_3' \sim \tau\mu\pi_2\pi_1'\pi_2'\pi_3'$

3. LOCALIZATION OF FUSION

Let G be a finite group and P a fixed Sylow p -subgroup of G .

Alperin introduced the concept of a conjugation family and proved that fusion of p -elements in G is controlled by the normalizers of certain p -Sylow intersections. The terminology and a proof of his main theorem may be found in Chapter 7 of [10].

Theorem 3.2 is due to Goldschmidt. A proof may essentially be found in [7].

Let \mathcal{F} be the set of all pairs (H, T) with

(1) $H = P \cap Q$, a tame Sylow intersection for some $Q \in \text{Syl } p(G)$ and $T = N_G(H)$ if H also satisfies

$$(2) \quad H = C_P(\Omega_1(Z(H)))$$

and

$$(3) \quad H \in \text{Syl } p(O_{p',p}(N_G(H)))$$

and

$$(4) \quad H = P \text{ or } N_G(H)/H \text{ is } p\text{-isolated;}$$

$$T = C_G(\Omega_1(Z(H))) \cap N_G(H) \quad \text{if} \quad H \text{ satisfies (3), (4)}$$

and

$$(2') \quad C_P(H) \subsetneq H < C_P(\Omega_1(Z(H)));$$

$$T = C_G(H) \quad \text{otherwise.}$$

LEMMA 3.1. \mathcal{F} is an inductive family.

Proof. By Theorem 3.3 of Goldschmidt [7] we may assume that the following situation obtains:

$H = P \cap Q$ is a tame Sylow intersection such that $R \sim_{\mathcal{F}} P$ for all $R \in \text{Syl } p(G)$ such that $|P \cap R| > |P \cap Q|$; and

$$C_P(H) \subseteq H < C_P(\Omega_1(Z(H))).$$

We must show that $Q \sim_{\mathcal{F}} R_0$ for some $R_0 \in \text{Syl } p(G)$ such that $|P \cap R_0| > |H|$.

Let

$$\begin{aligned} C_H &= C_G(\Omega_1(Z(H))) \cap N_G(H) \trianglelefteq N_G(H), \\ C_P &= C_H \cap N_P(H), \quad C_Q = C_H \cap N_Q(H). \end{aligned}$$

Since H is tame, $C_P, C_Q \in \text{Syl } p(C_H)$. Hence $\exists x \in C_H$ such that $C_Q^x = C_P$. As

$$C_P(\Omega_1(Z(H))) > H, \quad C_P = C_P(\Omega_1(Z(H))) \cap N_G(H) > H.$$

Hence

$$P \cap Q^x \geq C_Q^x = C_P > H \quad \text{and} \quad Q \sim_{\mathcal{F}} Q^x.$$

THEOREM 3.2. Suppose that G is a finite group, $P \in \text{Syl } p(G)$ and B_0 and B_1 are subsets of P which are conjugate in G . Then there exist subgroups H_1, H_2, \dots, H_n satisfying conditions (1), (3) and (4) above; elements $x_i \in N_G(H_i)$, $1 \leq i \leq n$, such that either (2) holds for H_i or (2') holds for H_i and $x_i \in C_G(\Omega_1(Z(H_i))) \cap N_G(H_i)$; and an element $y \in N_G(P)$ such that

- (a) $B_0 \subseteq H_1$;
- (b) $B_0^{x_1 x_2 \cdots x_i} \subseteq H_{i+1} \quad (1 \leq i \leq n-1)$;
- (c) $B_0^{x_1 x_2 \cdots x_n y} = B_1$.

Proof. This is immediate from Lemma 3.1.

We now specialize to the situation: G a finite group with Sylow 2-subgroup S of type \mathcal{U}_{12} .

We use the notation for elements and subgroups of S introduced in Chapter 2.

Under these hypotheses we prove

THEOREM 3.3. Suppose that B_0 and B_1 are subsets of S which are conjugate in G . Then there exist elements x_1, x_2, \dots, x_n with either $x_i \in N_G(A)$ or $x_i \in C_G(Z(S))$ such that

- (a) $B_0^{x_1 x_2 \cdots x_i} \subseteq S, \quad 1 \leq i \leq n$,
- (b) $B_0^{x_1 x_2 \cdots x_n} = B_1$.

We prove Theorem 3.3 by surveying those subgroups H of S which satisfy conditions (1)–(4) and

$$(5) \quad N_G(H) \not\subseteq N_G(A).$$

Condition (5) implies that $H \not\supseteq A$.

Together with condition (2), this implies that $H \subseteq C_S(x)$ for some involution $x \in S - A$. The distinct possibilities for $C_S(x)$ are

- (1) $H_1 = C_S(\mu) = \langle \pi_1, \pi_2, \pi_3, \pi_3' \rangle \langle \mu, \mu', \tau \rangle$,
- (2) $H_2 = C_S(\mu\pi_1) = \langle \pi_1, \pi_2, \pi_3, \pi_3' \rangle \langle \mu, \mu', \tau\pi_1'\pi_2' \rangle$,
- (3) $H_3 = C_S(\mu\pi_3') = \langle \pi_1, \pi_2, \pi_3, \pi_3' \rangle \langle \mu, \tau \rangle$,
- (4) $H_4 = C_S(\mu\pi_1\pi_3') = \langle \pi_1, \pi_2, \pi_3, \pi_3' \rangle \langle \mu, \tau\pi_1'\pi_2' \rangle$,
- (5) $H_5 = C_S(\mu') = \langle \pi_1, \pi_2, \pi_3, \pi_2' \rangle \langle \mu, \mu' \rangle$,
- (6) $H_6 = C_S(\mu'\mu) = \langle \pi_1, \pi_2, \pi_3, \pi_1' \rangle \langle \mu, \mu' \rangle$,
- (7) $H_7 = C_S(\mu'\pi_2') = \langle \pi_1, \pi_2, \pi_3, \pi_2', \mu' \rangle$,
- (8) $H_8 = C_S(\mu'\mu\pi_1') = \langle \pi_1, \pi_2, \pi_3, \pi_1', \mu'\mu \rangle$,
- (9) $H_9 = C_S(\tau) = \langle \pi_1\pi_2, \pi_3, \pi_1'\pi_2', \pi_3' \rangle \langle \mu, \tau \rangle$,
- (10) $H_{10} = C_S(\tau\pi_1'\pi_2') = \langle \pi_1\pi_2, \pi_3, \pi_1'\pi_2', \pi_3' \rangle \langle \mu\pi_1, \tau \rangle$,
- (11) $H_{11} = C_S(\tau\mu) = \langle \pi_1\pi_2, \pi_3, \pi_1\pi_1'\pi_2', \pi_3' \rangle \langle \mu, \tau \rangle$,
- (12) $H_{12} = C_S(\tau\mu\pi_1\pi_1'\pi_2') = \langle \pi_1\pi_2, \pi_3, \pi_1\pi_1'\pi_2', \pi_3' \rangle \langle \mu\pi_1, \tau\pi_1 \rangle$.

In particular, $H \neq S$, so $N_G(H)/H$ is 2-isolated. Let $N = N_G(H)/H$. Then, by Bender [1], either a Sylow 2-subgroup of N is cyclic or generalized quaternion or $N/O(N)$ contains a normal subgroup, N_1 , of odd index isomorphic to one of

- (i) $L_2(2^n) \quad n \geq 2$,
- (ii) $U_3(2^n) \quad n \geq 2$,
- (iii) $Sz(2^{2n+1}) \quad n \geq 1$.

As H is tame, $N_S(H)/H \in \text{Syl}_2(N)$. Also $|N_S(H)/H| \leq 2^5$ and N has a non-trivial $GF(2)$ -representation on $\Omega_1(Z(H))$ of dimension at most 5. It follows that either $N_S(H)/H$ is cyclic or generalized quaternion or $N/O(N) \cong L_2(4)$.

LEMMA 3.4. *Let K be an elementary abelian subgroup of S such that $K \not\subseteq A$ and either*

- (i) $K = C_S(K)$ or
- (ii) $|K| = 2^5$

holds. Then both (i) and (ii) hold. Moreover, $|N_S(K)/K| \geq 8$ and $N_S(K)/K$ contains a subgroup isomorphic to $Z_2 \times Z_2$.

Proof. Let $\bar{S} = S/A$. Then $\bar{K} \cong Z_2$ or $Z_2 \times Z_2$ and K splits over $K \cap A$, since K is elementary. Inspection of $H_1, H_2, H_5, H_6, H_9, H_{10}, H_{11}$ and

H_{12} reveals that $|C_A(x)| = 2^4$ for every involution $x \in S - A$ and $|C_A(\langle x, y \rangle)| = 2^3$ for every 4-group $\langle x, y \rangle$ contained in $S - A$.

Thus if either (i) or (ii) holds, then either $K = \langle C_A(x), x \rangle$ for some $x \in \{\mu, \mu', \mu'\mu, \tau, \tau\mu\}$ or $K = \langle C_A(\langle x, y \rangle), x, y \rangle$, where $\langle x, y \rangle$ is a 4-group contained in $S - A$.

Suppose that $\langle x, y \rangle = \langle \mu a, \mu' b \rangle$ for some $a \in C_A(\mu)$, $b \in C_A(\mu')$. Then

$$1 = [\mu a, \mu' b] = [\mu, b] [a, \mu'].$$

Since $[\mu, C_A(\mu')] = \langle \pi_2 \rangle$ and $[C_A(\mu), \mu'] = \langle \pi_3 \rangle$, $[\mu, b] = 1 = [\mu', a]$. Thus $\langle a, b \rangle \subseteq C_A(\langle \mu, \mu' \rangle)$ and $K = \langle C_A(\langle \mu, \mu' \rangle), \mu, \mu' \rangle$. Similarly for $\{\mu, \mu'\mu\}$ and $\{\mu', \mu'\mu\}$. If $\langle x, y \rangle = \langle \mu a, \tau b \rangle$, $a \in C_A(\mu)$, $b \in C_A(\tau)$, then

$$1 = [\mu a, \tau b] = [\mu, b] [a, \tau]$$

and either $1 = [\mu, b] = [a, \tau]$ or $\pi_1\pi_2 = [\mu, b] = [a, \tau]$. Thus either $K = \langle C_A(\langle \mu, \tau \rangle), \mu, \tau \rangle$ or $K = \langle C_A(\langle \mu\pi_1, \tau\pi_1'\pi_2' \rangle), \mu\pi_1, \tau\pi_1'\pi_2' \rangle$.

Consideration of $\{\mu, \tau\mu\}$ and $\{\tau, \tau\mu\}$ yields nothing new. Thus K is one of

- (i) $K_1 = \langle \pi_1, \pi_2, \pi_3, \pi_3', \mu \rangle$,
 $N_S(K_1) = S$, $C_S(K_1) = K_1$,
- (ii) $K_2 = \langle \pi_1, \pi_2, \pi_3, \pi_2', \mu' \rangle$,
 $N_S(K_2) = K_2 \langle \mu, \pi_1', \pi_3' \rangle$, $C_S(K_2) = K_2$,
- (iii) $K_3 = \langle \pi_1, \pi_2, \pi_3, \pi_1', \mu'\mu \rangle$,
 $N_S(K_3) = K_3 \langle \mu, \pi_2', \pi_3' \rangle$, $C_S(K_3) = K_3$,
- (iv) $K_4 = \langle \pi_1\pi_2, \pi_3, \pi_1'\pi_2', \pi_3', \tau \rangle$,
 $N_S(K_4) = K_4 \langle \mu, \pi_1, \pi_1' \rangle$, $C_S(K_4) = K_4$,
- (v) $K_5 = \langle \pi_1\pi_2, \pi_3, \pi_1\pi_1'\pi_2', \pi_3', \mu\tau \rangle$,
 $N_S(K_5) = K_5 \langle \mu, \pi_1, \pi_1' \rangle$, $C_S(K_5) = K_5$,
- (vi) $K_6 = \langle \pi_1, \pi_2, \pi_3, \mu, \mu' \rangle$,
 $N_S(K_6) = S$, $C_S(K_6) = K_6$,
- (vii) $K_7 = \langle \pi_1\pi_2, \pi_3, \pi_3', \mu, \tau \rangle$,
 $N_S(K_7) = K_7 \langle \pi_1, \pi_1'\pi_2', \mu' \rangle$, $C_S(K_7) = K_7$,
- (viii) $K_8 = \langle \pi_1\pi_2, \pi_3, \pi_3', \mu\pi_1, \tau\pi_1'\pi_2' \rangle$,
 $N_S(K_8) = K_8 \langle \pi_1, \pi_1'\pi_2', \mu' \rangle$, $C_S(K_8) = K_8$.

The lemma follows by inspection.

COROLLARY 3.5. $N_S(H)/H$ is cyclic.

Proof. Suppose that $N/O(N) \cong L_2(4)$. By condition (2), $N/O(N)$ acts nontrivially on $\Omega_1(Z(H))$, and by (2.2), $N/O(N)$ has at least three orbits on $\Omega_1(Z(H))^\#$. So $|\Omega_1(Z(H))| \geq 2^5$, whence $|\Omega_1(Z(H))| = 2^5$. As $\Omega_1(Z(H)) \not\subseteq A$ by (5), $\Omega_1(Z(H)) = K_i$ for some i , $1 \leq i \leq 8$. As $C_S(K_i) = K_i$ for all i , $H = \Omega_1(Z(H))$. But $|N_S(K_i)/K_i| \geq 8$ for all i , a contradiction.

Suppose that $N_S(H)/H$ is generalized quaternion. If $|\Omega_1(Z(H))| = 2^3$, then $\Omega_1(Z(H)) = \langle Z(S), x \rangle$, $x \in S - A$, and

$$|H| = |C_S(\Omega_1(Z(H)))| = |C_S(x)| \geq 2^5.$$

If $|\Omega_1(Z(H))| = 2^4$, then $H = C_S(\Omega_1(Z(H))) > \Omega_1(Z(H))$ by (3.4). So, in any case, $|H| \geq 2^5$ and $|N_S(H)| \geq 2^8$. Thus $|A \cap N_S(H)| \geq 2^5$.

Let $z \in N_S(H)$ such that \bar{z} is the involution in $N_S(H)/H$. Then $A \cap N_S(H) \subseteq H\langle z \rangle$. Hence $|H : A \cap H| = 2$. Since H must contain an involution of $S - A$ in its center, $H = \langle C_A(x), x \rangle$ for some involution $x \in S - A$. But then $N_S(H)/H$ contains a 4-group by (3.4), a contradiction. Thus $N_S(H)/H$ is cyclic.

LEMMA 3.6. $H = C_S(\mu)$ or $H = C_S(\mu\pi_1)$.

Proof. Suppose that $H < C_S(\mu)$. By condition (2),

$$H \supseteq Z(C_S(\mu)) = \langle \pi_1\pi_2, \pi_3, \mu \rangle = \Phi(C_S(\mu)).$$

So $C_S(\mu) \subseteq N_S(H)$ and $C_S(\mu)/H$ is elementary abelian, whence $[C_S(\mu) : H] = 2$. This forces $Z(H) \subseteq \langle \pi_1, \pi_2, \pi_3, \pi_3', \mu \rangle$, an abelian group. So $H \supseteq \langle \pi_1, \pi_2, \pi_3, \pi_3', \mu \rangle$. But then $\pi_1'\pi_2' \in N_S(H)$, contrary to $N_S(H)/H$ cyclic.

Similarly $H \triangleleft C_S(\mu\pi_1)$. In particular, $H \not\subseteq H_i$ for $i = 3$ or 4 . As H is not elementary by (3.4) and $|Z(H_i)| = 2^4$ for $i = 5, 6, \dots, 12$, if $H \subseteq H_i$ for $i = 5, 6, \dots, 12$, then $H = H_i$ and $i \neq 7$ or 8 . $N_S(H_5) = H_5\langle \pi_1', \pi_3' \rangle$ and $N_S(H_6) = H_6\langle \pi_2', \pi_3' \rangle$. So $H \neq H_i$ for $i = 5$ or 6 . Let $H = H_i$ for some $i \in \{9, 10, 11, 12\}$. Then $\pi_1 \notin H$, $\langle \pi_1\pi_2 \rangle = H'$ and $C = \{H \cap H' \cap 1\}$ is a critical chain of H stabilized by π_1 , contradicting Lemma 4.1 of Goldschmidt [7].

So $H = C_S(\mu)$ or $H = C_S(\mu\pi_1)$.

LEMMA 3.7. If H satisfies (1)–(5), then $N_G(H) \subseteq C_G(Z(S))$.

Proof. Let $H = C_S(\mu)$, $Z = \langle \pi_1\pi_2, \pi_3, \mu \rangle = Z(H) = \Phi(H)$. $Z\pi_1, Z\pi_3', Z\pi_1\pi_3', Z\mu', Z\mu'\pi_1, Z\tau, Z\tau\pi_3'$ are the nonidentity cosets of Z in H whose elements are involutions. Let λ be an element of odd order in $N_G(H)$. As $C_H(Z\pi_1)^\lambda = C_H(Z\pi_1)$, $Z\pi_1^\lambda \in \{Z\pi_1, Z\pi_3'\}$. If $Z\pi_1^\lambda = Z\pi_3'$, then $[Z\tau^\lambda, Z\pi_3'] = (\pi_1\pi_2)^\lambda$. But $[H, Z\pi_3'] = \langle \pi_3 \rangle$ and $\pi_1\pi_2 \not\sim_G \pi_3$ by (2.2). So $Z\pi_1^\lambda = Z\pi_1$ and $Z\pi_3'^\lambda = Z\pi_3'$. Then $Z\mu'^\lambda \in \{Z\mu', Z\mu'\pi_1\}$. If $Z\mu'^\lambda = Z\mu'\pi_1$, then $(Z\mu')^{\lambda^2} = Z\mu'$, contrary to the fact that λ has odd order. So $Z\mu'^\lambda = Z\mu'$ and λ acts trivially on $H/\Phi(H)$, hence on H .

As H is tame, $N_S(H) \in \text{Syl}_2(N_G(H))$. So $N_G(H) = N_S(H)C_G(H) \subseteq C_G(Z(S))$ for $H = C_S(\mu)$. As $\mu\pi_1 = \mu^{\pi_1'}$, $N_G(C_S(\mu\pi_1)) = N_G(C_S(\mu))^{\pi_1'} \subseteq C_G(Z(S))$ and we are done by (3.6).

We may now complete the proof of Theorem 3.3.

Consider the family \mathcal{F}_0 of all pairs (H, T) satisfying one of

I. H satisfies (1)–(4) and

$$T = N_G(H) \subseteq N_G(A) \quad \text{or} \quad T = N_G(H) \subseteq C_G(Z(S));$$

II. H satisfies (1), (2'), (3), (4),

$$T = N_G(H) \cap C_G(\Omega_1(Z(H))).$$

It follows from Lemmas 3.1 and 3.7 that \mathcal{F}_0 is a weak conjugation family. Moreover if H satisfies (2'), then $Z(S) \subseteq \Omega_1(Z(H))$. So

$$T = N_G(H) \cap C_G(\Omega_1(Z(H))) \subseteq N_G(H) \cap C_G(Z(S)).$$

The theorem follows directly from Theorem 3.2.

COROLLARY 3.8. *Suppose that $Z^*(G) = \langle 1 \rangle$. Then*

$$Z^*(N_G(A)) = O(N_G(A)).$$

Proof. Immediate from (3.3) and the Z^* -theorem.

4. THE FUSION IN $N_G(A)$

In this section, G is a finite group of type \mathcal{A}_{12} .

Notation.

$$\bar{N} = N_G(A)/C_G(A) \cong GL(6, 2);$$

$$\bar{D} \in \text{Syl}_2(\bar{N}) \text{ is the image of } D \text{ in } \bar{N}.$$

$$\tilde{N} = \bar{N}/O(\bar{N}); \quad \tilde{D} \text{ is the image of } D \text{ in } \tilde{N}.$$

By inspection in $GL(6, 2)$, we conclude

LEMMA 4.1. \tilde{N} is isomorphic to one of

- (i) D_8 (ii) \mathfrak{S}_4 (iii) \mathfrak{S}_5 (iv) $L_2(7)$ (v) \mathcal{A}_6 (vi) \mathcal{A}_7 .

LEMMA 4.2.

(a) $|O(\bar{N})|$ divides 3^4 .

(b) If \bar{T} is a 4-subgroup of \bar{D} , then $|C_{O(\bar{N})}(\bar{T})| \leq 3$.

(c) If \bar{N} is not 2-nilpotent, then $|C_{O(\bar{N})}(\bar{\mu})| = 1$ or 3 and $|O(\bar{N})| = 1, 3$ or 3^3 .

Proof. (a) With respect to the basis

$$\{\pi_1', \pi_1\pi_1', \pi_2', \pi_2\pi_2', \pi_3', \pi_3\pi_3'\} \quad \text{for } A,$$

$$\bar{\mu} \leftrightarrow \begin{pmatrix} 0 & 1 & & & & \\ & 1 & 0 & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix} \in GL(6, 2).$$

So $|C_{\bar{N}}(\bar{\mu})|_{2'}$ divides

$$\left| C_{GL(6,2)} \left(\begin{pmatrix} 0 & 1 & & & & \\ & 1 & 0 & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \end{pmatrix} \right) \right|_{2'} = 3^2.$$

Similarly $|C_{\bar{N}}(\bar{\mu}')|_{2'}$ divides 3^2 , as does $|C_{\bar{N}}(\bar{\mu}\bar{\mu}')|_{2'}$. So $|O(\bar{N})|$ divides $|GL(6, 2)|_3 = 3^4$.

(b) Suppose that \bar{T} centralizes a subgroup, \bar{V} , of $O(\bar{N})$ with $|\bar{V}| = 3^2$. By the $P \times Q$ -lemma applied to $\bar{T} \times \bar{V}$, \bar{V} acts faithfully on $C_A(\bar{T})$, a 3-dimensional subspace of A . But $|GL(3, 2)|_3 = 3$, a contradiction.

(c) Since \bar{N} is not 2-nilpotent, $\exists \bar{\rho} \in \bar{N}$ such that $\bar{T} = \langle \bar{\mu}, \bar{\mu}^{\bar{\rho}} \rangle$ is a 4-subgroup of \bar{D} .

Then

$$|O(\bar{N})| = \frac{|C_{O(\bar{N})}(\bar{\mu})|^3}{|C_{O(\bar{N})}(\bar{T})|^2}.$$

Moreover $3 \mid |\tilde{N}|$. So $|O(\bar{N})|$ divides 3^3 .

By (b), $|C_{O(\bar{N})}(\bar{T})| = 1$ or 3. So $|C_{O(\bar{N})}(\bar{\mu})| = 1$ or 3 and $|O(\bar{N})| = 1, 3$ or 3^3 .

COROLLARY 4.3. *If $|O(\bar{N})| = 3^3$, then $\tilde{N} \cong D_8$ or $\tilde{N} \cong \mathfrak{S}_4$. If $\bar{N} \cong \mathfrak{S}_4 = \mathfrak{S}_4 \cdot 3^3$, then*

$$O(\bar{N}) = C_{O(\bar{N})}(\bar{\mu}) \oplus C_{O(\bar{N})}(\bar{\mu}^{\bar{\rho}}) \oplus C_{O(\bar{N})}(\bar{\mu}^{\bar{\rho}^2}).$$

Proof. Suppose that $\tilde{N} \cong \mathfrak{S}_5$ or \tilde{N} is simple. Since neither \mathcal{O}_5 nor $L_2(7)$ is contained in $GL(3, 3)$, \tilde{N}' centralizes $O(\bar{N})$. But \tilde{N}' contains a 4-subgroup of \bar{D} , contradicting (4.2b).

Suppose that $\bar{N} \cong \tilde{\mathfrak{S}}_4$ and $O(\bar{N})$ is not abelian. Then $|Z(O(\bar{N}))| = 3$. Thus, by the structure of \bar{N} , $\langle \bar{\mu}, \bar{\mu}^\beta \rangle \subseteq \bar{N}' \subseteq C_{\bar{N}}(Z(O(\bar{N})))$. Then, by (4.2c), $C_{O(\bar{N})}(\bar{\sigma}) = Z(O(\bar{N}))$ for each $\bar{\sigma} \in \langle \bar{\mu}, \bar{\mu}^\beta \rangle^\#$. Hence

$$Z(O(\bar{N})) = \langle C_{O(\bar{N})}(\bar{\sigma}) \mid \bar{\sigma} \in \langle \bar{\mu}, \bar{\mu}^\beta \rangle^\# \rangle = O(\bar{N}),$$

a contradiction. Thus $O(\bar{N})$ is abelian and the conclusion follows.

LEMMA 4.4. *Suppose that $A = A_1 \times A_2$ with each A_i an indecomposable \bar{D} -subspace of A and $\dim A_1 \geq \dim A_2$. Then either $A_1 = \langle \pi_1, \pi_2, \pi_1'\pi_2', \pi_1' \rangle$ or $A_1 = \langle \pi_1, \pi_2, \pi_1'\pi_2', \pi_3\pi_1' \rangle$ and either $A_2 = \langle \pi_3, \pi_3' \rangle$ or $A_2 = \langle \pi_3, \pi_1\pi_2\pi_3' \rangle$.*

Proof. $A = \langle \pi_1, \pi_2, \pi_1', \pi_2' \rangle \times \langle \pi_3, \pi_3' \rangle = \tilde{A}_1 \times \tilde{A}_2$ is a decomposition of the required type.

So if $A = A_1 \times A_2$ is any such decomposition, $A_i \cong \tilde{A}_i$ as \bar{D} -modules, for $i = 1, 2$, by the Krull-Schmidt theorem. So

$$A_2 \subseteq C_A(\langle \bar{\mu}, \bar{\tau} \rangle) = \langle \pi_1\pi_2, \pi_3, \pi_3' \rangle \quad \text{and} \quad C_{A_2}(\bar{\mu}') \neq A_2.$$

So $A_2 = \langle \pi_3, \pi_3' \rangle$ or $A_2 = \langle \pi_3, \pi_1\pi_2\pi_3' \rangle$. Then $a\pi_1' \in A_1$ for some $a \in A_2$. So $(a\pi_1')^\tau = a\pi_2' \in A_1$, whence $A_1 = \langle \pi_1, \pi_2, \pi_1'\pi_2', a\pi_1' \rangle$. Since $\pi_3 \notin A_1$, $a = 1$ or $a = \pi_3$.

LEMMA 4.5. *Suppose that \bar{N} acts indecomposably on A .*

(a) $\bar{\mu}' \sim_{\bar{N}} \bar{\mu}$.

(b) \bar{N} is isomorphic to one of

(i) \mathfrak{S}_4 (ii) $\tilde{\mathfrak{S}}_4$ (iii) \mathfrak{S}_5 (iv) $L_2(7)$ (v) \mathcal{O}_6 (vi) \mathcal{O}_7 .

Proof. (a) As $\langle \bar{\mu}' \rangle$ is a vertex for A_1 as a \bar{D} -module and $\langle \bar{\mu}, \bar{\tau} \rangle$ is a vertex for A_2 as a \bar{D} -module, $\langle \bar{\mu}, \bar{\tau} \rangle$ must be a vertex for A as an \bar{N} -module and $\langle \bar{\mu}' \rangle \subseteq_N \langle \bar{\mu}, \bar{\tau} \rangle$ [3, p. 161]. By Alperin's theorem, $N_{\bar{N}}(\langle \bar{\mu}, \bar{\mu}' \rangle)$ and $N_{\bar{N}}(\langle \bar{\mu}, \bar{\tau} \rangle)$ control fusion of involutions in \bar{D} with respect to \bar{N} . Hence $\langle \bar{\mu}' \rangle \subseteq_N \langle \bar{\mu}, \bar{\tau} \rangle$ implies $\bar{\mu}' \sim_{\bar{N}} \bar{\mu}$.

(b) If \bar{N} is 2-nilpotent, $\bar{\mu}' \not\sim_{\bar{N}} \bar{\mu}$; so \bar{N} acts decomposably on A by part (a).

Suppose that \bar{N} is not 2-nilpotent and that $|O(\bar{N})| = 3$. By the structure of \bar{N} , $O(\bar{N}) \subseteq C_{\bar{N}}(\bar{T})$ for some 4-subgroup, \bar{T} , of \bar{D} . Hence $O(\bar{N})$ normalizes $C_A(\bar{T})$, a group of order 8; whence $O(\bar{N})$ has a fixed point on $A^\#$. Then $A = C_A(O(\bar{N})) \times [A, O(\bar{N})]$ is a nontrivial decomposition of A into \bar{N} -subspaces.

The conclusion now follows from (4.1) and (4.3).

LEMMA 4.6. *If $\bar{N} \cong \tilde{\mathfrak{S}}_4$ and \bar{N} acts indecomposably on A , then \bar{N} is uniquely determined up to isomorphism.*

Proof. Since $\tilde{\mathfrak{S}}_4$ acts indecomposably on A , $\langle \bar{\mu}, \bar{\mu}' \rangle \subseteq \bar{N}'$ and, by (4.3),

$$O(\bar{N}) = C_{O(\bar{N})}(\bar{\mu}) \oplus C_{O(\bar{N})}(\bar{\mu}\bar{\mu}') \oplus C_{O(\bar{N})}(\bar{\mu}').$$

Now $\bar{D} \cdot O(\bar{N})$ acts decomposably on A . If $O(\bar{N}) = [O(\bar{N}), \langle \bar{\mu}, \bar{\tau} \rangle]$, then $O(\bar{N})$ centralizes A_2 . But then $A = C_A(O(\bar{N})) \times [A, O(\bar{N})]$ is a nontrivial decomposition of A into \bar{N} -subspaces. So

$$\langle 1 \rangle < C_{O(\bar{N})}(\langle \bar{\mu}, \bar{\tau} \rangle) \subseteq C_{O(\bar{N})}(\bar{\mu}),$$

whence $C_{O(\bar{N})}(\langle \bar{\mu}, \bar{\tau} \rangle) = C_{O(\bar{N})}(\bar{\mu})$. Let $\bar{\xi} \in N_{\bar{N}}(\langle \bar{\mu}, \bar{\mu}' \rangle)$ such that $\bar{\mu}^{\bar{\xi}} = \bar{\mu}'\bar{\mu}$, and $\bar{\xi}^{\bar{\tau}} = \bar{\xi}^{-1}$. Let

$$\bar{x}_1 \in C_{O(\bar{N})}(\bar{\mu}'\bar{\mu}), \quad \bar{x}_2 = \bar{x}_1^{\bar{\xi}}, \quad \bar{x}_3 = \bar{x}_1^{\bar{\xi}^2}.$$

Then

$$\begin{aligned} O(\bar{N}) &= \langle \bar{x}_1 \rangle \oplus \langle \bar{x}_2 \rangle \oplus \langle \bar{x}_3 \rangle. \\ \bar{x}_1^{\bar{\mu}} &= \bar{x}_1^{\bar{\mu}'} = \bar{x}_1^{-1}, & \bar{x}_2^{\bar{\mu}} &= \bar{x}_2^{-1}, & \bar{x}_2^{\bar{\mu}'} &= \bar{x}_2, \\ \bar{x}_3^{\bar{\mu}} &= \bar{x}_3^{\bar{\tau}} = \bar{x}_3, & \bar{x}_3^{\bar{\mu}'} &= \bar{x}_3^{-1}, \\ \bar{x}_1^{\bar{\tau}} &= \bar{x}_3^{\bar{\xi}\bar{\tau}} = \bar{x}_3^{\bar{\tau}\bar{\xi}^{-1}} = \bar{x}_3^{\bar{\xi}^{-1}} = \bar{x}_2. \end{aligned}$$

Since a Sylow 3-subgroup of \bar{N} is isomorphic to $Z_3 \wr Z_3$, \bar{N} splits over $O(\bar{N})$ by a theorem of Gaschütz [5].

We assume henceforth that $Z^*(G) = \langle 1 \rangle$.

THEOREM 4.7. *If \bar{N} acts indecomposably on A , then \bar{N} is isomorphic to one of*

- (i) $\tilde{\mathfrak{S}}_4$,
- (ii) $L_2(7)$,
- (iii) \mathcal{O}_7 acting with the 6-dimensional irreducible permutation action.

Proof. I. Suppose that $\bar{N} \cong \mathfrak{S}_4$. Then \bar{N} normalizes $\bar{T} = \langle \bar{\mu}, \bar{\mu}' \rangle$; hence normalizes $C_A(\bar{T})$, a group of order 8. But then, since an \mathfrak{S}_3 cannot act fixed point freely on an E_8 , \bar{N} must fix an element of $C_A(\bar{T})^\#$, contrary to (3.8).

II. Suppose that $\bar{N} \cong \mathfrak{S}_5$, \mathcal{O}_6 or \mathcal{O}_7 and case (iii) does not hold. The only other nontrivial irreducible $GF(2)$ -modules for \mathfrak{S}_5 , \mathcal{O}_6 or \mathcal{O}_7 of dimension not exceeding 6, have dimension 4. Since $Z(\bar{N} \cdot A) = \langle 1 \rangle$, this implies that A contains a 4-dimensional irreducible \bar{N} -submodule, V , and $\langle \bar{\mu}, \bar{\mu}' \rangle$ acts trivially on A/V . So

$$V \supseteq [A, \langle \bar{\mu}, \bar{\mu}' \rangle] = \langle \pi_1, \pi_2, \pi_3 \rangle \supseteq Z(S).$$

But \bar{N} contains an \mathcal{O}_5 ; hence has at most two orbits on $V^\#$, contradicting (2.2).

THEOREM 4.8. *According as (i), (ii) or (iii) of Theorem 4.7 holds, the fusion under G of the involutions of A (under a suitable renaming of the generators of A) is*

- (i) 1-2-3-4 | 5-6-7-8-9-10-11 | 12-13-14-15-16-17-18,
- (ii) 1-2-3-4-10-11 | 5-6-18 | 7-8-9-15-16-17 | 12-13-14,
- (iii) 1-2-3-4-10-11 | 5-6-7-8-9-15-16-17-18 | 12-13-14.

Proof. By Gaschütz [5], \bar{N} splits over A . Also $\langle \bar{\mu}, \bar{\tau} \rangle$ is a vertex for A as \bar{N} -module and $A_{\bar{H}} = A_1 \oplus A_2$, the A_i uniquely determined indecomposable \bar{D} -modules.

- (i) $\bar{N} \cong \tilde{\mathfrak{S}}_4$.

We apply Green's theory with $P = \langle \bar{\mu}, \bar{\tau} \rangle$,

$$H = N_{\bar{N}}(P) = (\langle \bar{\mu}, \bar{\tau} \rangle \times \langle \bar{x}_3 \rangle) \langle \bar{\mu}' \rangle \quad \text{and} \quad R = GF(2).$$

By the $P \times Q$ -lemma applied to $\langle \bar{\mu}, \bar{\tau} \rangle \times \langle \bar{x}_3 \rangle$, $\langle \bar{x}_3 \rangle$ acts nontrivially on A_2 . So A_2 is a uniquely determined $R[H]$ -module with vertex $\langle \bar{\mu}, \bar{\tau} \rangle$. Since $\langle \bar{\mu}, \bar{\tau} \rangle = P$, Green's theorem guarantees the existence and uniqueness of an indecomposable $R[\bar{N}]$ -module, V , with vertex $\langle \bar{\mu}, \bar{\tau} \rangle$, such that $A_2 | V_H$. (See [3, Chapter III, Section5].)

So $\bar{N} \cdot A$ is a uniquely determined split extension. If E is an elementary group of order 2^6 in \mathcal{O}_{12} , then $\bar{N} \cdot A \cong N_{\mathcal{O}_{12}}(E)$ and the fusion of involutions of A in G is that of involutions of E in \mathcal{O}_{12} (see Yamaki [17]).

- (ii) $\bar{N} \cong L_2(7)$.

Let $\bar{\xi} \in \bar{N}$ of order 3 such that $\bar{\mu}^{\bar{\xi}} = \bar{\mu}'\bar{\mu}$, $\bar{\mu}^{\bar{\xi}^2} = \bar{\mu}'$. Then $\bar{\xi}$ normalizes $C_A(\langle \bar{\mu}, \bar{\mu}' \rangle) = \langle \pi_1, \pi_2, \pi_3 \rangle$ and permutes

$$[A, \bar{\mu}] = \langle \pi_1, \pi_2 \rangle, \quad [A, \bar{\mu}'\bar{\mu}] = \langle \pi_2, \pi_3 \rangle \quad \text{and} \quad [A, \bar{\mu}'] = \langle \pi_3, \pi_1 \rangle.$$

So $\bar{\xi}$ centralizes α . Since $Z(\bar{N} \cdot A) = \langle 1 \rangle$ and $N_{\bar{N}}(\langle \bar{\mu}, \bar{\mu}' \rangle)$ is a maximal subgroup of \bar{N} , $N_{\bar{N}}(\langle \bar{\mu}, \bar{\mu}' \rangle) = C_{\bar{N}}(\alpha)$.

Let $\bar{\omega} \in \bar{N}$ of order 3 such that $\bar{\mu}^{\bar{\omega}} = \bar{\tau}\bar{\mu}$, $\bar{\mu}^{\bar{\omega}^2} = \bar{\tau}$. Then $\bar{\omega}$ normalizes

$$C_A(\langle \bar{\mu}, \bar{\tau} \rangle) \cap [A, \langle \bar{\mu}, \bar{\tau} \rangle] = \langle \pi_1\pi_2, \pi_3, \pi_3' \rangle \cap \langle \pi_1, \pi_2, \pi_1'\pi_2' \rangle = \langle \pi_1\pi_2 \rangle.$$

So $\bar{\omega}$ centralizes $\pi_1\pi_2$. As above, this implies that $N_{\bar{N}}(\langle \bar{\mu}, \bar{\tau} \rangle) = C_{\bar{N}}(\pi_1\pi_2)$. Let $P = \langle \bar{\mu}, \bar{\tau} \rangle$, $H = N_{\bar{N}}(P)$ and $R = GF(2)$. Since $\bar{\mu}' \not\sim_H \bar{\mu}$, H acts decomposably on A . As H centralizes $\pi_1\pi_2$, $C_H(\pi_3) = C_H(\alpha) = \bar{D}$. So A_2 is a uniquely determined $R[H]$ -module with vertex $\langle \bar{\mu}, \bar{\tau} \rangle$.

Again Green's theorem gives the existence and uniqueness of an indecomposable $R[\bar{N}]$ -module, V , with vertex $\langle \bar{\mu}, \bar{\tau} \rangle$ such that $A_2 \mid V_H$.

So $\bar{N} \cdot A$ is a uniquely determined split extension. If E is an elementary group of order 2^6 in $\text{Sp}_6(2)$, then $\bar{N} \cdot A \cong N \text{Sp}_6(2) (E)$ and the fusion of involutions of A in G is that of involutions of E in $\text{Sp}_6(2)$ (see Yamaki [18]).

(iii) $\bar{N} \cong \mathcal{O}_7$.

By (4.7), the action of \bar{N} is given by the unique 6-dimensional irreducible $GF(2)$ -representation. So the split extension, $\bar{N} \cdot A$, and the fusion in A are uniquely determined.

We regard \mathcal{O}_7 as acting on $\{e_1 + e_2, e_1 + e_3, \dots, e_1 + e_7\}$ by permutation of the subscripts. With respect to the basis

$$B = \{\pi_1', \pi_1\pi_1', \pi_2', \pi_2\pi_2', \pi_3', \pi_3\pi_3'\},$$

$$\bar{\mu} \leftrightarrow (23) \quad (45) \quad \bar{\mu}' \leftrightarrow (23) \quad (67) \quad \bar{\tau} \leftrightarrow (24) \quad (35).$$

So B is an admissible basis for the representation.

$\langle (1234567) \rangle$ applied to $e_2 + e_3$ gives

$$\pi_1 \sim \pi_1\pi_1'\pi_2' \sim \pi_2\pi_2'\pi_3' \sim \pi_3 \sim \pi_3\pi_3' \sim \pi_1',$$

i.e.,

$$1 \sim 2 \sim 3 \sim 4 \sim 10 \sim 11.$$

$\langle (1234567) \rangle$ applied to $e_2 + e_3 + e_4 + e_5$ gives

$$\pi_1\pi_2 \sim \pi_1\pi_2\pi_1'\pi_3' \sim \pi_2\pi_3 \sim \pi_2\pi_3\pi_2' \sim \pi_3\pi_1' \sim \pi_1\pi_3\pi_3' \sim \pi_1\pi_2',$$

i.e.

$$5 \sim 6 \sim 7 \sim 8 \sim 9 \sim 15.$$

$\langle (1234567) \rangle$ applied to $e_2 + e_3 + e_4 + e_5 + e_6 + e_7$ gives

$$\alpha \sim \alpha\pi_1' \sim \alpha\pi_3', \quad \text{i.e.,} \quad 12 \sim 13 \sim 14.$$

$\langle (123) \rangle$ applied to $e_1 + e_2 + e_4 + e_6$ gives

$$\alpha' \sim \pi_1\pi_2'\pi_3' \sim \pi_1\alpha', \quad \text{i.e.,} \quad 15 \sim 17 \sim 18.$$

Finally

$$(e_2 + e_3 + e_4 + e_6) (367) = e_2 + e_4 + e_6 + e_7, \quad \text{i.e.,} \quad 15 \sim 16.$$

LEMMA 4.9. In cases (i) and (ii) of Theorem 4.8, $C_{\bar{N}}(Z(S)) \cong D_8$. In case (iii), $C_{\bar{N}}(Z(S)) \cong \mathfrak{S}_4$ and

$$(1)_{C_{\bar{N}}(Z(S))} \widetilde{(10)}, (6)_{C_{\bar{N}}(Z(S))} \widetilde{(16)}, (8)_{C_{\bar{N}}(Z(S))} \widetilde{(17)}_{C_{\bar{N}}(Z(S))} \widetilde{(18)},$$

all other S -classes of A unfused in $C_G(Z(S))$.

Proof. The result for cases (i) and (ii) is clear by inspection. In case (iii), an element of $C_{\bar{N}}(Z(S))$ must fix the sets $\{2, 3, 4, 5\}$ and $\{6, 7\}$. Hence $C_{\bar{N}}(Z(S)) \cong \mathfrak{S}_4$ and (234) is a typical 3-element of $C_{\bar{N}}(Z(S))$. The fusion is readily verified.

COROLLARY 4.10. *There exists a subgroup, C_0 , of $C_G(Z(S))$ of index 2 with $A\langle\mu, \tau\rangle \in \text{Syl}_2(C_0)$.*

Proof. By (4.9) and the fact that \bar{N} normalizes A_2 in the decomposable case, $\{\pi_3', \pi_3\pi_3'\}$ is a full $C_G(Z(S))$ -class of A . As each of the classes (19)–(21), (24)–(28) is closed under multiplication by $\pi_1\pi_2$ and as $\pi_3' \not\sim_G \pi_1\pi_2\pi_3'$, no one of these classes is fused to π_3' in $C_G(Z(S))$. Thus $\{\pi_3', \pi_3\pi_3'\}$ is a full $C_G(Z(S))$ -class of $C_S(\pi_3')$. But then if $\mu'a, a \in C_A(\mu')$, is fused to π_3' in $C_G(Z(S))$, $\exists g \in C_G(Z(S))$ sending $\mu'a$ to π_3' and $C_S(\mu'a)$ into $C_S(\pi_3')$; hence sending $\{\mu'a, \mu'a\pi_1, \mu'a\pi_3, \mu'a\pi_1\pi_3\}$ into $\{\pi_3', \pi_3\pi_3'\}$, an impossibility.

So $\bar{\pi}_3' \in Z^*(C_G(Z(S))/Z(S))$. Hence

$$\langle \bar{\pi}_1\bar{\pi}_2, \bar{\pi}_3, \bar{\pi}_3' \rangle \trianglelefteq C_G(Z(S))/O(C_G(Z(S))) = \bar{C}$$

and

$$C_{\bar{C}}(\langle \bar{\pi}_1\bar{\pi}_2, \bar{\pi}_3, \bar{\pi}_3' \rangle) = \bar{C}_0$$

has index 2 in \bar{C} and $\bar{A}\langle\bar{\mu}, \bar{\tau}\rangle \in \text{Syl}_2(\bar{C}_0)$. So

$$O(C_G(Z(S))) C_{C_G(Z(S))}(\pi_3') = C_0$$

has index 2 in $C_G(Z(S))$ and $A\langle\mu, \tau\rangle \in \text{Syl}_2(C_0)$.

THEOREM 4.11. *If $G = O^2(G)$, then \bar{N} acts indecomposably on A . Hence $\bar{N} \cong \mathfrak{S}_4, L_2(7)$ or \mathcal{O}_7 with the action of \bar{N} on A as described in Theorem 4.8.*

Proof. Suppose that \bar{N} acts decomposably on $A = A_1 \times A_2$. Let $N = N_G(A)$. Then $C_N(A_2)$ is normal in N of index 6 and

$$A\langle\mu, \tau\rangle \in \text{Syl}_2(C_N(A_2)).$$

Suppose that μ' is fused into $A\langle\mu, \tau\rangle$ in G . By (3.3), $\exists x_1, \dots, x_n$ with $x_i \in N$ or $x_i \in C_G(Z(S))$ for $1 \leq i \leq n$ such that

$$(i) \quad \mu'^{x_1 \dots x_i} \in S \text{ for all } i,$$

$$(ii) \quad \mu'^{x_1 \dots x_n} \in A\langle\mu, \tau\rangle.$$

Then $\exists j, 1 < j \leq n$, such that

$$\mu'^{x_1 \dots x_{j-1}} \in S - A\langle\mu, \tau\rangle \quad \text{and} \quad (\mu'^{x_1 \dots x_{j-1}})^{x_j} \in A\langle\mu, \tau\rangle.$$

But either $x_j \in N$ or $x_j \in C_G(Z(S))$ and $A\langle\mu, \tau\rangle = S \cap C_N(A_2) = S \cap C_0$, a contradiction in either case.

So μ' is not fused in G into the maximal subgroup $A\langle\mu, \tau\rangle$ of S . But then $O^2(G) < G$ by Thompson's transfer lemma.

5. A STRONGLY CLOSED ABELIAN 2-GROUP

In this section we prove

THEOREM 5.1. *Let G be a fusion-simple group with Sylow 2-subgroups of type \mathcal{O}_{12} . Let $S \in \text{Syl}_2(G)$ and let A be the unique elementary abelian subgroup of S of order 2^6 . If $N_G(A)$ controls fusion in S with respect to G , then $G = N_G(A)$.*

By (4.11) and Glauberman's theorem, we may assume that $\bar{N} \cong L_2(7)$ or $\bar{N} \cong \mathcal{O}_7$.

We shall prove that $N_G(A)$ is a weakly embedded subgroup of G . Then, by Goldschmidt [9], $G = N_G(A)$.

Notation. Fix for the remainder of the paper elements $\xi \in N_G(A)$ in cases (i), (ii) and (iii) and $\omega \in N_G(A)$ in cases (ii) and (iii) only, both of odd order and acting as follows on the generators of S :

	ξ	ω
π_1	π_2	$\pi_1\pi_1'\pi_2'$
π_2	π_3	$\pi_2\pi_1'\pi_2'$
π_3	π_1	π_3'
π_1'	π_2'	$\pi_1\pi_3\pi_1'\pi_3'$
π_2'	π_3'	$\pi_3\pi_1'\pi_3'$
π_3'	π_1'	$\pi_3\pi_3'$
μ	$\mu\mu'$	$\mu\tau$
μ'	μ	$\mu'\omega$
τ	τ^ξ	μ

LEMMA 5.2. *If $\bar{N} \cong L_2(7)$, then $S - A$ has 3 G -classes of involutions with representatives μ , $\mu\pi_3$, $\mu\pi_3'$.*

If $\bar{N} \cong \mathcal{O}_7$, then $S - A$ has 2 G -classes of involutions with representatives μ and $\mu\pi_3$.

Proof. $\mu'^\xi = \mu$; $(\mu'\pi_2)^\xi = \mu\pi_3$; $(\mu'\pi_2')^\xi = \mu\pi_3'$.

$\tau^\omega = \mu$; $(\tau\pi_3)^\omega = \mu\pi_3'$; $(\tau\pi_3')^\omega = \mu\pi_3\pi_3'$; $(\tau\pi_3\pi_3')^\omega = \mu\pi_3$. In the notation of (4.8), case (iii), the permutation (176) fixes μ and sends π_3 to π_3' ; hence sends $\mu\pi_3$ to $\mu\pi_3'$. Let $na \in \bar{N} \cdot A$, $n \in \bar{N}$, $a \in A$, such that $(\mu\beta)^{na} = \mu\gamma$, with

$\beta, \gamma \in C_A(\mu)$. By the uniqueness of expression in $\bar{N} \cdot A$, $n \in C_{\bar{N}}(\mu)$ and $\mu\gamma = \mu^a\beta^n$. Thus if $\beta = 1$, $\mu\gamma = \mu^a$, $a \in A$; whence $\mu\pi_3 \not\sim_G \mu \not\sim_G \mu\pi_3'$. So if $\bar{N} \cong \mathcal{O}_7$, $S - A$ has exactly 2 G -classes of involutions. If $\bar{N} \cong L_2(7)$, $C_{\bar{N}}(\mu) = \bar{D}$. So $na \in S$ and $\mu\pi_3 \not\sim_G \mu\pi_3'$ in this case.

Case A. $\bar{N} \cong L_2(7)$.

LEMMA 5.3. $C_G(\pi_3) = S \cdot O(C_G(\pi_3))$.

Proof. By (4.8), π_3 has 21 G -conjugates in A . Hence

$$N_G(A) \cap C_G(\pi_3) = C_G(A) \cdot D.$$

Thus $N_{C_G(\pi_3)}(A)$ is 2-nilpotent; whence $C_G(\pi_3)$ is 2-nilpotent by Glauberman's theorem.

LEMMA 5.4. $C_G(\alpha)/O(C_G(\alpha))$ is covered by $N_G(A)$.

Proof. By (4.8),

$$C_G(\alpha) \cap N_G(A) = C_G(A) N_{N_G(A)}(\langle \mu, \mu' \rangle) = C_G(A) \langle \mu, \mu', \tau, \xi \rangle.$$

Thus $\tau \notin C_G(\alpha)'$ by Glauberman's theorem.

Let $\bar{C} = C_G(\alpha)/\langle \alpha \rangle \times O(C_G(\alpha))$.

$$\begin{aligned} \bar{\pi}_2^{\bar{\xi}} &= \bar{\pi}_3; & \bar{\pi}_2'^{\bar{\xi}} &= \bar{\pi}_3'; & (\bar{\pi}_1\bar{\pi}_2')^{\bar{\xi}} &= \bar{\pi}_2\bar{\pi}_3'; & (\bar{\pi}_1'\bar{\pi}_2')^{\bar{\xi}} &= \bar{\pi}_2'\bar{\pi}_3'; \\ \bar{\mu}^{\bar{\xi}} &= \bar{\mu}'\bar{\mu}; & \text{and} & & (\bar{\mu}\bar{\pi}_3')^{\bar{\xi}} &= \bar{\mu}'\bar{\mu}\bar{\pi}_1'. \end{aligned}$$

Thus the involution fusion pattern in $\bar{A}\langle\bar{\mu}, \bar{\mu}'\rangle$ with respect to \bar{C} is

$$\bar{1}-\bar{2} \mid \bar{3}-\bar{4} \mid \bar{7}-\bar{8} \mid \bar{10}-\bar{11} \mid \bar{17} \mid \bar{19}-\bar{22} \mid \bar{21}-\bar{24}.$$

By (2.1), $N_{\bar{C}}(\bar{S}) = \bar{S}C_{\bar{C}}(\bar{S})$. So, by Grün's theorem, $\exists \bar{C}_0 \trianglelefteq \bar{C}$ of index 4 with

$$\begin{aligned} \bar{S}_0 &= \langle (\bar{S}')^{N_{\bar{C}}(\bar{A})} \cap \bar{S} \rangle = \langle \{\bar{S}'\}^{\langle \bar{\xi} \rangle} \cap \bar{S} \rangle \\ &= \langle \bar{\pi}_1\bar{\pi}_2, \bar{\pi}_2\bar{\pi}_3, \bar{\pi}_1'\bar{\pi}_2', \bar{\pi}_2'\bar{\pi}_3', \bar{\mu}, \bar{\mu}' \rangle \in \text{Syl}_2(\bar{C}_0). \end{aligned}$$

Let $\bar{A}_0 = \langle \bar{\pi}_1\bar{\pi}_2, \bar{\pi}_2\bar{\pi}_3, \bar{\pi}_1'\bar{\pi}_2', \bar{\pi}_2'\bar{\pi}_3' \rangle$ and let $\bar{B}_0 = \langle \bar{\pi}_1\bar{\pi}_2, \bar{\pi}_2\bar{\pi}_3, \bar{\mu}, \bar{\mu}' \rangle$. By (5.3),

$$C_{\bar{C}_0}(\bar{\pi}_1\bar{\pi}_2) = C_{\bar{C}_0}(\bar{\pi}_3) = O(C_{\bar{C}_0}(\bar{\pi}_1\bar{\pi}_2)) \bar{S}_0.$$

As $\bar{\pi}_1\bar{\pi}_2$ has exactly 3 conjugates in \bar{A}_0 and in \bar{B}_0 ,

$$N_{\bar{C}_0}(\bar{A}_0) = C_{\bar{C}_0}(\bar{A}_0) \langle \bar{S}_0, \bar{\xi} \rangle \quad \text{and} \quad N_{\bar{C}_0}(\bar{B}_0) = C_{\bar{C}_0}(\bar{B}_0) \langle \bar{S}_0, \bar{\xi} \rangle.$$

Now every involution of \bar{S}_0 lies in either \bar{A}_0 or \bar{B}_0 . Also $\bar{\xi}$ acts fixed point freely on both $\bar{A}_0^\#$ and $\bar{B}_0^\#$. So $C_{\bar{C}_0}(\bar{\sigma}) \cap N_{\bar{C}_0}(\bar{A}_0)$ and $C_{\bar{C}_0}(\bar{\sigma}) \cap N_{\bar{C}_0}(\bar{B}_0)$ are 2-nilpotent for every involution $\bar{\sigma} \in \bar{S}_0$.

Since \bar{A}_0 is strongly closed in \bar{S}_0 with respect to \bar{C}_0 , $C_{\bar{C}_0}(\bar{\sigma})$ is 2-nilpotent for $\bar{\sigma} \in \bar{A}_0^\#$ by Glauberman's theorem.

Since \bar{B}_0 is abelian and $C_{\bar{S}_0}(\bar{\sigma}) = \bar{B}_0$ for every $\bar{\sigma} \in \bar{B}_0 - \bar{A}_0$, $C_{\bar{C}_0}(\bar{\sigma})$ is 2-nilpotent for $\bar{\sigma} \in \bar{B}_0 - \bar{A}_0$ by Burnside's theorem.

So $C_{\bar{C}_0}(\bar{\sigma})$ is 2-nilpotent for every involution $\bar{\sigma} \in \bar{S}_0$. As \bar{C}_0 has more than one class of involutions, $\bar{C}_0 = \bar{S}_0 \langle \bar{\xi} \rangle$.

LEMMA 5.5. $C_G(\pi_1\pi_2)/O(C_G(\pi_1\pi_2))$ is covered by $N_G(A)$.

Proof. By (4.8),

$$C_G(\pi_1\pi_2) \cap N_G(A) = C_G(A) N_{N_G(A)}(\langle \mu, \tau \rangle) = C_G(A) \langle \mu, \mu', \tau, \omega \rangle.$$

Thus $\mu' \notin C_G(\pi_1\pi_2)'$ by Glauberman's theorem.

$$\begin{aligned} \pi_1^\omega &= \pi_1\pi_1'\pi_2'; & \pi_3^\omega &= \pi_3'; & \pi_1'^\omega &= \pi_1\pi_3\pi_1'\pi_3'; \\ (\pi_1\pi_3)^\omega &= \pi_1\alpha'; & (\pi_1\pi_2')^\omega &= \pi_1\pi_3\pi_2'\pi_3'; & (\pi_1\pi_3')^\omega &= \pi_1\pi_3\alpha'; \\ (\pi_1\pi_3\pi_3')^\omega &= \pi_1\pi_3\pi_1'\pi_2'; & (\pi_2\pi_3\pi_2')^\omega &= \pi_2\pi_3\pi_2'; & \mu^\omega &= \tau\mu; \\ (\mu\pi_3)^\omega &= \tau\mu\pi_3'; & (\mu\pi_3')^\omega &= \tau\mu\pi_3\pi_3'; & \text{and} & (\mu\pi_3\pi_3')^\omega &= \tau\mu\pi_3. \end{aligned}$$

Thus the involution fusion pattern in $A\langle \mu, \tau \rangle$ with respect to $C_G(\pi_1\pi_2)$ is
 1-10 | 2-4 | 3-11 | 5 | 6-18 | 7-15 | 8-16-17 | 9 | 12-13 | 14 | 19-25 | 20-28 |
 21-26-27.

By (2.1), $N_C(S) = SC_C(S)$. So, by Grün's theorem, $\exists C_0 \trianglelefteq C$ of index 4 with

$$\begin{aligned} S_0 &= \langle (S')^{N_{C_G(\pi_1\pi_2)}(A)} \cap S \rangle = \langle (S')^{\langle \omega \rangle} \cap S \rangle \\ &= \langle \pi_1, \pi_3, \pi_1'\pi_2', \pi_3', \mu, \tau \rangle \in \text{Syl}_2(C_0). \end{aligned}$$

Let $\bar{C} = C_G(\pi_1\pi_2)/\langle \pi_1\pi_2 \rangle \times O(C_G(\pi_1\pi_2))$. Then $\bar{C}_0 \trianglelefteq \bar{C}$ of index 4; \bar{S}_0 is abelian; and, by (5.3) $N_{\bar{C}_0}(\bar{S}_0) = C_{\bar{C}_0}(\bar{S}_0) \langle \bar{\omega} \rangle$ with $\bar{\omega}$ acting fixed point freely on $\bar{S}_0^\#$. So $\bar{C}_0 = \bar{S}_0 \langle \bar{\omega} \rangle$ by Walter [16].

LEMMA 5.6. $C_G(\alpha')/O(C_G(\alpha'))$ is covered by $N_G(A)$.

Proof. By (4.8), α' has 28 G -conjugates in A . So

$$N_{C_G(\alpha')}(A) = C_G(A) \langle \tau, \xi \rangle.$$

Thus $\alpha \in Z^*(C_G(\alpha'))$, whence $C_G(\alpha') \subseteq O(C_G(\alpha')) C_G(\alpha)$. The lemma follows by (5.4).

LEMMA 5.7. $C_G(\mu a)/O(C_G(\mu a))$ is covered by $N_G(A)$ for $a \in \{1, \pi_3\}$.

Proof. By (4.8), $\{\pi_1, \pi_2, \pi_3, \pi_3', \pi_3\pi_3'\}$ is a full G -class of $C_S(\mu a)$. Since μa is extremal, $C_S(\mu a) \in \text{Syl}_2(C_G(\mu a))$. As $\mu a \pi_1 \not\sim_G \mu a \pi_3 \not\sim_G \mu a \pi_3'$, $\pi_1 \not\sim_{C_G(\mu a)} \pi_3 \not\sim_{C_G(\mu a)} \pi_3'$. So $\pi_3 \in Z^*(C_G(\mu a))$, whence

$$C_G(\mu a) \subseteq O(C_G(\mu a)) C_G(\pi_3).$$

The lemma follows by (5.3).

LEMMA 5.8. $C_G(\mu\pi_3')/O(C_G(\mu\pi_3'))$ is covered by $N_G(A)$.

Proof. As $\mu\pi_1\pi_3' \not\sim_G \mu \not\sim_G \mu\pi_3\pi_3'$ and $\mu \not\sim_G \mu\pi_3$,

$$\pi_1 \not\sim_{C_G(\mu\pi_3')} \pi_3' \not\sim_{C_G(\mu\pi_3')} \pi_3 \quad \text{and} \quad \pi_3' \not\sim_{C_G(\mu\pi_3')} \pi_3\pi_3'.$$

So $\pi_3' \in Z^*(C_G(\mu\pi_3'))$, whence $C_G(\mu\pi_3') \subseteq O(C_G(\mu\pi_3')) C_G(\pi_3')$. As $\pi_3' \sim_{N_G(A)} \pi_3$, the lemma follows by (5.3).

Since $\{\pi_3, \pi_1\pi_2, \alpha, \alpha', \mu, \mu\pi_3, \mu\pi_3'\}$ is a set of representatives for the $N_G(A)$ -classes of involutions, $N_G(A)$ is weakly embedded in G . So Theorem 5.1 holds in this case.

COROLLARY 5.9. $G = A\langle\mu, \mu', \tau, \xi, \omega\rangle = \bar{N} \cdot A$.

Case B. $\bar{N} \cong \mathcal{A}_7$.

LEMMA 5.10. $C_G(\pi_3)/O(C_G(\pi_3))$ is covered by $N_G(A)$.

Proof. In the notation of (4.8), $\pi_3 \leftrightarrow e_6 + e_7$ in the permutation representation of \bar{N} on A . Thus

$$C_{\bar{N}}(\pi_3) = \mathcal{U}_{\{1,2,3,4,5\}} \cdot \langle \bar{\mu}' \rangle.$$

$$\pi_1 \leftrightarrow e_2 + e_3 \underset{(314)}{\sim} \pi_1' \leftrightarrow e_1 + e_2 \underset{(143)}{\sim} \pi_1'\pi_2' \leftrightarrow e_2 + e_4.$$

Since $\mathcal{U}_{\{1,4,5\}}\langle \bar{\mu}, \bar{\mu}' \rangle$ centralizes π_1 , $\{1-3-10\}$ is a full $C_G(\pi_3)$ -class of A .

$$\pi_3' \leftrightarrow e_1 + e_6 \underset{(123)}{\sim} \pi_1'\pi_3' \leftrightarrow e_2 + e_6.$$

Since $\mathcal{U}_{\{2,3,4,5\}}$ centralizes π_3' , $\{4-11\}$ is a full $C_G(\pi_3)$ -class of A .

$$\pi_1\pi_2 \leftrightarrow e_2 + e_3 + e_4 + e_5 \underset{(231)}{\sim} \pi_1\pi_2\pi_1' \leftrightarrow e_1 + e_3 + e_4 + e_5.$$

Since $\mathcal{O}_{\{2,3,4,5\}}\langle\bar{\mu}'\rangle$ centralizes $\pi_1\pi_2$, $\{5-7\}$ is a full $C_G(\pi_3)$ -class of A .

$$\begin{aligned}\pi_1\pi_3' &\leftrightarrow e_1 + e_2 + e_3 + e_6 \underset{(124)}{\sim} \pi_1\pi_2'\pi_3' \leftrightarrow e_2 + e_3 + e_4 + e_6 \underset{(341)}{\sim} \alpha' \\ &\leftrightarrow e_1 + e_2 + e_4 + e_6 \underset{(243)}{\sim} \pi_1\alpha' \leftrightarrow e_1 + e_3 + e_4 + e_6.\end{aligned}$$

Since $\mathcal{O}_{\{1,2,3\}}\langle\bar{\mu}\rangle$ centralizes $\pi_1\pi_3'$, $\{8-15-17-18\}$ is a full $C_G(\pi_3)$ -class of A . $\pi_1\pi_2\pi_3' \leftrightarrow e_1 + e_2 + e_3 + e_4 + e_5 + e_6$ is centralized by $\mathcal{O}_{\{1,2,3,4,5\}}$. Thus $\pi_1\pi_2\pi_3'$ is $C_{\bar{N}}(\pi_3)$ -conjugate only to $\alpha\pi_3'$.

Since A is strongly closed in S with respect to $C_G(\pi_3)$, it follows that $\exists C_0 \trianglelefteq C_G(\pi_3)$ of index 2 with $A\langle\bar{\mu}, \bar{\tau}\rangle \in \text{Syl}_2(C_0)$ and $\pi_1\pi_2\pi_3' \in Z^*(C_0)$.

Let $\bar{C}_0 = C_0/Z^*(C_0)$. Then $\bar{S}_0 = \langle\bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2'\rangle\langle\bar{\mu}, \bar{\tau}\rangle \in \text{Syl}_2(\bar{C}_0)$. Since

$\langle\bar{\mu}, \bar{\tau}\rangle \subseteq \mathcal{O}_{\{1,2,3,4,5\}}$, $\bar{\mu} \sim_{\bar{C}_0} \bar{\tau}$. Thus the involution fusion pattern in \bar{C}_0 is

$$\bar{\pi}_1\bar{\pi}_2 \sim \bar{\pi}_1\bar{\pi}_2\bar{\pi}_1' \mid \bar{\pi}_1 \sim \bar{\pi}_1' \sim \bar{\pi}_1'\bar{\pi}_2' \mid \bar{\mu} \sim \bar{\tau}.$$

So \bar{C}_0 has the involution fusion pattern of $\mathcal{O}_5 \cdot E_{16}^{(1)}$; whence $C_0 \cong \mathcal{O}_5 \cdot E_{16}^{(1)}$ by Proposition 3.4 of Gorenstein–Harada [12].

LEMMA 5.11. $C_G(\pi_1\pi_2)/O(C_G(\pi_1\pi_2))$ is covered by $N_G(A)$.

Proof. In the notation of (4.8), $\pi_1\pi_2 \leftrightarrow e_2 + e_3 + e_4 + e_5$ in the permutation representation of \bar{N} on A . Thus

$$\begin{aligned}C_{\bar{N}}(\pi_1\pi_2) &= (\mathcal{O}_{\{2,3,4,5\}} \times \mathcal{O}_{\{1,6,7\}}) \cdot \langle\bar{\mu}'\rangle. \\ \pi_1 &\leftrightarrow e_2 + e_3 \underset{(324)}{\sim} \pi_1'\pi_2' \leftrightarrow e_2 + e_4.\end{aligned}$$

Since $\mathcal{O}_{\{1,6,7\}}\langle\bar{\mu}, \bar{\mu}'\rangle = C_{\bar{N}}(\pi_1, \pi_2)$, $\{1-10\}$ is a full $C_G(\pi_1\pi_2)$ -class of A .

$$\pi_3 \leftrightarrow e_6 + e_7 \underset{(761)}{\sim} \pi_3' \leftrightarrow e_1 + e_6.$$

Since $\mathcal{O}_{\{2,3,4,5\}}\langle\bar{\mu}'\rangle = C_{\bar{N}}(\pi_1\pi_2, \pi_3)$, $\{2-4\}$ is a full $C_G(\pi_1\pi_2)$ -class of A .

$$\pi_1' \leftrightarrow e_1 + e_2 \underset{(167)}{\sim} \pi_1'\pi_3' \leftrightarrow e_2 + e_6.$$

Since $\mathcal{O}_{\{3,4,5\}}\langle\bar{\mu}'\bar{\mu}\rangle = C_{\bar{N}}(\pi_1\pi_2, \pi_1')$, $\{3-11\}$ is a full $C_G(\pi_1\pi_2)$ -class of A .

$$\begin{aligned}\pi_1\pi_3 &\leftrightarrow e_2 + e_3 + e_6 + e_7 \underset{(324)}{\sim} \pi_3\pi_1'\pi_2' \leftrightarrow e_2 + e_4 + e_6 + e_7 \underset{(671)}{\sim} \pi_3\alpha' \\ &\leftrightarrow e_1 + e_2 + e_4 + e_7 \underset{(423)}{\sim} \pi_1\pi_3\pi_3' \leftrightarrow e_1 + e_2 + e_3 + e_7 \underset{(234)}{\sim} \pi_1\pi_3\alpha' \\ &\leftrightarrow e_1 + e_3 + e_4 + e_7.\end{aligned}$$

Since $\langle\bar{\mu}, \bar{\mu}'\rangle = C_{\bar{N}}(\pi_1\pi_2, \pi_1\pi_3)$, $\{6-8-16-17-18\}$ is a full $C_G(\pi_1\pi_2)$ -class of A .

$\pi_3\pi_1' \leftrightarrow e_1 + e_2 + e_6 + e_7$ is centralized by $\langle(345)\rangle \times \langle(167)\rangle$, a Sylow 3-subgroup of $C_N(\pi_1\pi_2)$.

Finally, $\mu \leftrightarrow (23) (45)$ and $\tau \leftrightarrow (24) (35)$. So

$$\begin{aligned}\mu^{(243)} &= \tau; & (\mu\pi_3)^{(243)} &= \tau\pi_3; & (\mu\pi_3)^{(176)} &= \mu\pi_3'; \\ (\mu\pi_3)^{(243)(176)} &= \tau\pi_3'; & \text{and} & & (\mu\pi_3)^{(243)(167)} &= \tau\pi_3\pi_3' .\end{aligned}$$

Thus $\mu' \notin (S')^{C_N(\pi_1\pi_2)}$.

By the above, the involution fusion pattern in $A\langle\mu, \tau\rangle$ with respect to $C_G(\pi_1\pi_2)$ is

$$\begin{aligned}1-10 \mid 2-4 \mid 3-11 \mid 5 \mid 6-8-16-17-18 \mid 7-15 \mid 9 \mid 12-13 \mid 14 \mid 19-25 \mid \\ 20-21-26-27-28.\end{aligned}$$

Let $\bar{C} = C_G(\pi_1\pi_2)/\langle\pi_1\pi_2\rangle \times O(C_G(\pi_1\pi_2))$. As $N_{\bar{C}}(\bar{S}) = \bar{S}C_{\bar{C}}(\bar{S})$, $\exists \bar{C}_0 \trianglelefteq \bar{C}$ of index 4, by Grün's theorem, with

$$\bar{S}_0 = \langle(\bar{S}')^g \cap \bar{S} \mid g \in \bar{C}\rangle = \langle\bar{\pi}_1, \bar{\pi}_3, \bar{\pi}_1'\bar{\pi}_2', \bar{\pi}_3', \bar{\mu}, \bar{\tau}\rangle \in \text{Syl}_2(\bar{C}_0).$$

Since \bar{S}_0 is abelian, $O^{2'}(\bar{C}_0)$ is a direct product of elementary components and simple components by Walter [16].

Now, by 5.10 and the above,

$$C_G(\pi_3, \pi_1\pi_2) = O(C_G(\pi_3, \pi_1\pi_2)) \langle S, \lambda \rangle,$$

where λ corresponds to the permutation (243) in the action of N on A . So

$$C_{\bar{C}_0}(\bar{\pi}_3) = \langle\bar{\pi}_3, \bar{\pi}_3'\rangle \times \langle\bar{\pi}_1, \bar{\pi}_1'\bar{\pi}_2'\rangle \langle\bar{\mu}, \bar{\tau}, \bar{\lambda}\rangle \cong D_4 \times D_4 \cdot \mathcal{O}_4.$$

As $\bar{\pi}_3$ has exactly 3 conjugates in \bar{S}_0 with respect to \bar{C}_0 , $N_{\bar{C}_0}(\bar{S}_0) = \bar{S}_0\langle\bar{\omega}, \bar{\lambda}\rangle$.

Suppose that \bar{L} is a simple component of $O^{2'}(\bar{C}_0)$. Every element of \bar{S}_0 has either 3 or 9 conjugates in \bar{S}_0 with respect to \bar{C}_0 . Thus $\bar{L} \cong L_2(5)$. A 3-element of $N_{\bar{L}}(\bar{S}_0)$ must centralize a 4-dimensional subspace of \bar{S}_0 . Hence $\bar{\omega}\bar{\lambda}\bar{s} \in \bar{L}$ for some $\bar{s} \in \bar{S}_0$ and $O^{2'}(\bar{C}_0) = \bar{L} \times \bar{F}$ with $\langle\bar{\pi}_1, \bar{\pi}_1'\bar{\pi}_2', \bar{\mu}, \bar{\tau}\rangle \in \text{Syl}_2(\bar{F})$. But

$$C_G(\pi_1)/O(C_G(\pi_1)) = (\langle\bar{\pi}_1, \bar{\pi}_1'\rangle \times \langle\bar{\pi}_2, \bar{\pi}_2', \bar{\pi}_3, \bar{\pi}_3'\rangle \cdot \bar{K}_1) \langle\bar{\mu}\rangle,$$

where $\bar{K}_1 \cong \mathcal{O}_5$ and $C_{\langle\pi_2, \pi_2', \pi_3, \pi_3'\rangle}(\bar{K}_1) = \langle\bar{1}\rangle$. So $C_G(\pi_1, \pi_2)$ is solvable.

Hence so is $C_{\bar{C}}(\bar{\pi}_1) = \overline{C_G(\pi_1, \pi_2)} \langle\bar{\tau}\rangle$. But $\bar{L} \subseteq C_{\bar{C}_0}(\bar{\pi}_1)$, a contradiction.

So $O^{2'}(\bar{C}_0) = \bar{S}_0$, whence $\bar{C} = \bar{S}_0\langle\bar{\omega}, \bar{\tau}, \bar{\pi}_1', \bar{\mu}'\rangle \subseteq N_{\bar{C}}(\bar{A})$. Thus $N_G(A)$ covers $C_G(\pi_1\pi_2)/O(C_G(\pi_1\pi_2))$ as claimed.

LEMMA 5.12. $C_G(\mu)/O(C_G(\mu))$ is covered by $N_G(A)$.

Proof. By (4.8), $\{5-6-8\}$ is a full G -class of $C_S(\mu)$. As

$$\mu\pi_1\pi_3 \not\sim_G \mu\pi_1\pi_2 \not\sim_G \mu\pi_1\pi_3', \pi_1\pi_3 \not\sim_{C_G(\mu)} \pi_1\pi_2 \not\sim_{C_G(\mu)} \pi_1\pi_3'.$$

So $\pi_1\pi_2 \in Z^*(C_G(\mu))$ and we are done by (5.11).

LEMMA 5.13. $C_G(\mu\pi_3)/O(C_G(\mu\pi_3))$ is covered by $N_G(A)$.

Proof. As $\mu\pi_1\pi_3 \not\sim_G \mu \not\sim_G \mu\pi_3\pi_3'$, $\pi_1 \not\sim_{C_G(\mu\pi_3)} \pi_3 \not\sim_{C_G(\mu\pi_3)} \pi_3'$. So $\pi_3 \in Z^*(C_G(\mu\pi_3))$ and we are done by (5.10).

LEMMA 5.14. $C_G(\alpha)/O(C_G(\alpha))$ is covered by $N_G(A)$.

Proof. Let $C = C_G(\alpha)$. In the notation of (4.8), $\alpha \leftrightarrow e_2 + e_3 + e_4 + e_5 + e_6 + e_7$ in the permutation representation of \bar{N} on A . Hence $C \cap \bar{N} = \mathcal{O}_{\{2,3,4,5,6,7\}}$.

$C_{\bar{N}}(\pi_3) \cap C = \mathcal{O}_{\{2,3,4,5\}} \cdot \langle \bar{\mu}' \rangle$. So π_3 has 15 conjugates in A with respect to C . Since \mathcal{O}_6 is 2-transitive,

$$\pi_3 \leftrightarrow e_6 + e_7 \sim_C \pi_1 \leftrightarrow e_2 + e_3 \sim_C \pi_1'\pi_2' \leftrightarrow e_2 + e_4 \sim_C \pi_2'\pi_3' \leftrightarrow e_4 + e_6.$$

So $\{1-2-10-11\}$ is a full C -class of A .

Also $\pi_3' \leftrightarrow e_1 + e_6 \sim_C \pi_1' \leftrightarrow e_1 + e_2$. Hence, by (4.8), $\{3-4\}$ is a full C -class of A .

Since $C_{\bar{N}}(\pi_1\pi_2') \cap C = (\mathcal{O}_{\{2,3,4\}} \times \mathcal{O}_{\{5,6,7\}}) \cdot \langle \bar{\mu}' \rangle$, $\pi_1\pi_2'$ has 20 conjugates in A with respect to C .

$$\begin{aligned} \pi_1\pi_2' \leftrightarrow e_1 + e_2 + e_3 + e_4 &\stackrel{(23467)}{\sim} \pi_1\alpha' \leftrightarrow e_1 + e_3 + e_4 + e_6 \stackrel{(23467)}{\sim} \pi_3\pi_2' \\ &\leftrightarrow e_1 + e_4 + e_6 + e_7 \stackrel{(24736)}{\sim} \pi_1\pi_3\pi_3' \leftrightarrow e_1 + e_2 + e_3 + e_7 \stackrel{(32476)}{\sim} \alpha' \\ &\leftrightarrow e_1 + e_2 + e_4 + e_6. \end{aligned}$$

Thus $\{7-8-9-17-18\}$ is a full C -class of A .

Now $\mu \leftrightarrow (23) (45)$, $\mu' \leftrightarrow (23) (67)$ and $\tau \leftrightarrow (24) (35)$. So

$$(\mu\pi_3')^{(46)(57)} = \mu'\pi_2'; \quad (\mu\pi_3')^{(34)(67)} = \tau\pi_3\pi_3';$$

and

$$(\mu\pi_3\pi_3')^{(34)(67)} = \tau\pi_3'.$$

Let $\bar{C} = C/\langle \alpha \rangle \times O(C)$.

By the above, the involution fusion pattern in \bar{S} with respect to \bar{C} is

$$\overline{1-2-10-11} \mid \overline{3-4} \mid \overline{7-8-9-17-18} \mid \overline{19-22-25} \mid \overline{21-24-27-28}.$$

If σ is an involution in S not G -conjugate to α , then

$$C_G(\sigma) = O(C_G(\sigma)) N_{C_G(\sigma)}(A).$$

$\overline{C_G(\sigma)}$ has index 1 or 2 in $C_G(\bar{\sigma})$. Thus if $\bar{\sigma}$ is an extremal conjugate in \bar{S} , $C_G(\bar{\sigma}) \subseteq O(C_G(\bar{\sigma})) N_{C_G(\bar{\sigma})}(\bar{A})$. Now $\{\pi_3, \pi_3', \alpha', \mu, \mu\pi_3'\}$ is a set of involutions of S such that $\sigma \not\sim_G \alpha$ for $\sigma \in \{\pi_3, \pi_3', \alpha', \mu, \mu\pi_3'\}$ and $\{\bar{\pi}_3, \bar{\pi}_3', \bar{\alpha}', \bar{\mu}, \bar{\mu}\bar{\pi}_3'\}$ is a set of extremal representatives for the classes of involutions in \bar{C} . As $N_{C_G(\bar{A})}$ controls fusion in \bar{C} , it follows that $N_{C_G(\bar{A})}$ is weakly embedded in \bar{C} . Hence $\bar{C} = N_{C_G(\bar{A})}$ by Goldschmidt [9].

As $\{\pi_3, \pi_1\pi_2, \alpha, \mu, \mu\pi_3\}$ is a set of representatives for the $N_G(A)$ -classes of involutions, $N_G(A)$ is weakly embedded in G .

So Theorem 5.1 holds in all cases.

COROLLARY 5.15. *Under the hypotheses of Theorem 5.1, either $G = A\langle\mu, \mu', \tau, \xi, \omega\rangle$ or $G \cong \mathcal{O}_{7^+} \cdot E_{64}$.*

6. THE MAIN THEOREM

By Theorem 3.3 and the results of Section 5, we may now assume that some involutions $\sigma \in S - A$ is fused into A by an element of $C_G(Z(S))$.

Moreover by Corollary 4.10, $\sigma \in A\langle\mu, \tau\rangle - A$.

LEMMA 6.1. *Every involution of $A\langle\mu, \tau\rangle$ is fused into A by an element of $C_G(Z(S))$.*

Proof. For all involutions $\sigma \in A\langle\mu, \tau\rangle - A$, $\sigma \sim_S \sigma\pi_1\pi_2$. Thus if $\sigma^g \in A$, $g \in C_G(Z(S))$, $(\sigma\pi_1\pi_2)^g \sim_{C_G(Z(S))} \sigma^g$. By Lemma 4.9, then, $\sigma^g \in (1), (6), (8), (10), (16), (17)$ or (18) . So we may assume that $\sigma \mapsto \sigma^g$ is an extremal conjugation and $C_S(\sigma)^g \subseteq C_S(\sigma^g)$.

In particular, as $\pi_3' \in C_S(\sigma)$, $\pi_3'^g \in \{\pi_3', \pi_3\pi_3'\}$. Thus μ is fused into A in $C_G(Z(S))$ if and only if μa is fused into A for all $a \in C_A(\mu)$ and τ is fused into A in $C_G(Z(S))$ if and only if τb is fused into A for all $b \in C_{A\langle\mu\rangle}(\tau)$.

Suppose that μ is not fused into A in $C_G(Z(S))$ but $\tau^g \in A$ for some $g \in C_G(Z(S))$. Again assume that the conjugation is extremal. Then $\mu^g \in A\mu$. But then $\tau^g \sim_{C_G(Z(S))} (\tau\mu)^g \in A\mu$, a contradiction. Thus since σ is fused into A in $C_G(Z(S))$ for some involution $\sigma \in A\langle\mu, \tau\rangle - A$, μ is. Let $g \in C_G(Z(S))$ such that $\mu^g \in A$, μ^g extremal, $C_S(\mu)^g \subseteq C_S(\mu^g)$. Since $\pi_3 \in C_S(\mu)'$, $\pi_3 \in C_S(\mu^g)'$; hence $\mu^g \in (1)$ or (6) . Since $[\mu^g, \tau^g] = [\mu, \tau] = 1$, $\tau^g \in A\langle\mu\rangle$. So every involution of $A\langle\mu, \tau\rangle$ is fused into A in $C_G(Z(S))$.

Notation. Let $\xi, \omega \in N_G(A)$ be as in Lemma 5.2.

In case (i) of Theorem 4.7, we fix elements $x_i \in N_G(A)$, $1 \leq i \leq 3$, with $\pi_j, \pi_j' \in C_A(x_i)$ for $i \neq j$, $\pi_i^{x_i} = \pi_i'$, $\pi_i^{x_i'} = \pi_i \pi_i'$ (see Lemma 4.6).

In case (i), $\bar{N} = \langle \bar{\mu}, \bar{\mu}', \bar{\tau}, \bar{\xi}, \bar{x}_1, \bar{x}_2, \bar{x}_3 \rangle$.

In case (ii), $\bar{N} = \langle \bar{\mu}, \bar{\mu}', \bar{\tau}, \bar{\xi}, \bar{\omega} \rangle$.

THEOREM 6.2. *According as case (i), (ii) or (iii) of (4.8) holds, the fusion pattern for involutions of S in G (under a suitable renaming of generators, if necessary) is*

(i) 1-2-3-4-19-22 | 5-6-7-8-9-10-11-20-21-23-24-25 | 12-13-14-15-16-17-18-26-27-28.

(ii) 1-2-3-4-10-11-19-22-25 | 5-6-18-20-23-28 | 7-8-9-15-16-17-21-24-26-27 | 12-13-14.

(iii) 1-2-3-4-10-11-19-22-25 | 5-6-7-8-9-15-16-17-18-20-21-23-24-26-27-28 | 12-13-14.

Proof. Choose $g \in C_G(Z(S))$ such that $\mu \mapsto \mu^g \in A$ is an extremal conjugation and $C_S(\mu)^g \subseteq C_S(\mu^g)$. Then $\mu^g \in (1)$ or (6) .

Replacing g by $g\mu'$, if necessary, we may assume that $\pi_3'^g = \pi_3'$.

Replacing μ, τ by $\mu\pi_3, \tau\pi_3'$, resp., if necessary, we may assume that $\mu^g \in (1)$.

Then $(\mu\pi_3)^g \in (6)$ and $(\mu\pi_3')^g \in (8)$. Now $\mu'^\varepsilon = \mu$, $(\mu'\pi_2)^\varepsilon = \mu\pi_3$ and $(\mu'\pi_2')^\varepsilon = \mu\pi_3'$. In cases (ii) and (iii), $\tau^\omega = \mu$; $(\tau\pi_3)^\omega = \mu\pi_3'$, $(\tau\pi_3')^\omega = \mu\pi_3\pi_3'$ and $(\tau\pi_3\pi_3')^\omega = \mu\pi_3$.

So the determination of the fusion of involutions in S in cases (ii) and (iii) is complete. In case (i) it remains to determine τ^g . Since $(\tau\pi_1\pi_2)^g$ is in the same $C_G(Z(S))$ -class as τ^g , $\tau^g \in (1), (6), (8), (10), (16), (17), (18), (19), (20)$ or (21) . Since $[\tau, \mu'] = \mu$, $[\tau^g, \mu'^g] = \pi_1$ or π_2 . Thus since $\mu'^g \in (C_{A\langle\mu\rangle}(\mu')) \mu'$, $\tau^g \in (10)$ or (16) . Interchanging τ and $\tau\pi_3$, if necessary, we may assume that $\tau^g \in (10)$. Then $(\tau\pi_3)^g \in (16)$, $(\tau\pi_3')^g \in (17)$ or (18) and $(\tau\pi_3\pi_3')^g \in (17)$ or (18) . Since $(17) \sim_G (18)$, the fusion is determined in this case as well. We note that in case (iii) G has the involution fusion pattern of $\Omega_7(q)$, $q \equiv \pm 3 \pmod{8}$. Henceforth we assume that either case (i) or case (ii) holds.

LEMMA 6.3. $C_G(Z(S))/Z(S) \times O(C_G(Z(S))) \cong (\mathfrak{S}_4 \cdot E_8^{(2)}) \times Z_2$, where $\mathfrak{S}_4 \cdot E_8^{(2)}$ is the subgroup of $GL(3,2) \cdot E_8$ which normalizes one 4-subgroup of E_8 .

Proof. By (4.10), $\exists C_0 \trianglelefteq C_G(Z(S))$ of index 2 with $A\langle\mu, \tau\rangle \in \text{Syl}_2(C_0)$. By (4.9), (6.1) and (6.2), no involution of $A\langle\mu, \tau\rangle = \langle \pi_1, \pi_2, \pi_3, \pi_1'\pi_2', \pi_3', \mu, \tau \rangle$ is fused into $\langle \pi_1, \pi_2, \pi_3, \pi_1'\pi_2', \pi_3', \mu, \tau \rangle$ in $C_G(Z(S))$. So, by Thompson's transfer lemma, $\exists C_1 \trianglelefteq C_0$ of index 2 with

$$\langle \pi_1, \pi_2, \pi_3, \pi_1'\pi_2', \pi_3', \mu, \tau \rangle \in \text{Syl}_2(C_1).$$

By (4.10), $O(C_G(Z(S))) \langle \pi_1 \pi_2, \pi_3, \pi_3' \rangle \trianglelefteq C_G(Z(S))$. Let

$$\bar{C} = C_G(Z(S))/O(C_G(Z(S))) \langle \pi_1 \pi_2, \pi_3, \pi_3' \rangle$$

and let \bar{C}_1 be the image of C_1 in \bar{C} . Then $\bar{S}_1 = \langle \bar{\pi}_1, \bar{\pi}_1' \bar{\pi}_2', \bar{\mu}, \bar{\tau} \rangle \in \text{Syl}_2(\bar{C}_1)$. As $C_G(Z(S))$ is not 2-nilpotent by (6.1), it follows from (4.9) and (6.2) that $\exists \bar{\rho} \in N_{\bar{C}_1}(\bar{S}_1)$ acting fixed-point freely on $\bar{S}_1^\#$ and that each element of $\bar{S}_1^\#$ has exactly 3 conjugates in \bar{S}_1 with respect to \bar{C}_1 . Then, by Lemma 4.2 of Gorenstein–Harada [11], $|\bar{C}_1| = 3 \cdot 2^4$.

So $\bar{C} = \langle \bar{\pi}_1, \bar{\pi}_1' \bar{\pi}_2', \bar{\mu}, \bar{\tau} \rangle \langle \bar{\mu}', \bar{\pi}_2' \bar{\pi}_3', \bar{\rho} \rangle$. Since $\bar{\rho}$ does not normalize $\langle \bar{\pi}_1 \rangle = C_{\bar{S}_1}(\langle \bar{\mu}', \bar{\pi}_2' \bar{\pi}_3' \rangle)$, $\bar{\rho}$ does not normalize $\langle \bar{\mu}', \bar{\pi}_2' \bar{\pi}_3' \rangle$. So $\langle \bar{\mu}', \bar{\pi}_2' \bar{\pi}_3' \rangle$ normalizes a Sylow 3-subgroup of \bar{C} , which we may take to be $\langle \bar{\rho} \rangle$. As $\langle \bar{\rho} \rangle$ does not normalize $\langle \bar{\pi}_1, \bar{\pi}_1' \bar{\pi}_2' \rangle = C_{\bar{S}}(\bar{\pi}_2' \bar{\pi}_3')$, $\bar{\pi}_2' \bar{\pi}_3'$ inverts $\langle \bar{\rho} \rangle$ and $\bar{\mu}'$ centralizes $\langle \bar{\rho} \rangle$. Also $\bar{\rho}$ normalizes $\langle \bar{\pi}_1, \bar{\mu} \rangle$ and $\langle \bar{\pi}_1' \bar{\pi}_2', \bar{\tau} \rangle$. So

$$\bar{C} = \langle \bar{\pi}_1, \bar{\mu}, \bar{\mu}' \rangle \langle \bar{\pi}_1' \bar{\pi}_2', \bar{\pi}_2' \bar{\pi}_3', \bar{\tau}, \bar{\rho} \rangle \cong \mathfrak{S}_4 \cdot E_8^{(2)}.$$

As a Sylow 2-subgroup of $\tilde{C} = C_G(Z(S))/Z(S) \times O(C_G(Z(S)))$ splits over $\langle \bar{\pi}_3' \rangle$, \tilde{C} splits over $\langle \bar{\pi}_3' \rangle$ by Gaschütz [5].

Let $\hat{C} = C_G(Z(S))/O(C_G(Z(S)))$. Choose $\hat{\rho}$ of order 3 in the preimage of $\bar{\rho}$. Then $\hat{\mu}'^{\hat{\rho}} = \hat{\mu}' \hat{z}$ for some $\hat{z} \in Z(\hat{S})$. So $\hat{\mu}'^{\hat{\rho}^2} = \hat{\mu}'$; whence $\hat{z} = \hat{1}$ and $\hat{\rho} \in C_{\hat{C}}(\hat{\mu}')$. Thus we may select $\hat{\rho}$ in

$$N_{\hat{C}}(\langle \hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_1' \hat{\pi}_2', \hat{\pi}_3', \hat{\mu}, \hat{\mu}', \hat{\tau} \rangle) \cap C_{\hat{C}}(\langle \hat{\mu}', \hat{\pi}_3' \rangle).$$

By (6.2) and the above,

$$\hat{\tau}^{\hat{\rho}} \in \{ \hat{\pi}_1' \hat{\pi}_2', \hat{\tau} \hat{\pi}_1' \hat{\pi}_2', \hat{\pi}_1 \hat{\pi}_2 \hat{\pi}_1' \hat{\pi}_2', \hat{\tau} \hat{\pi}_1 \hat{\pi}_2, \hat{\tau} \hat{\pi}_1 \hat{\pi}_2 \hat{\pi}_1' \hat{\pi}_2' \}.$$

Since $(\hat{\tau} \hat{\pi}_1 \hat{\pi}_2)^{\hat{\rho}} = \hat{\tau}^{\hat{\rho}} \hat{\pi}_1 \hat{\pi}_2$, we may assume, by replacing $\hat{\rho}$ by $\hat{\rho}^2$ if necessary, that $\hat{\tau}^{\hat{\rho}} \in \{ \hat{\pi}_1' \hat{\pi}_2', \hat{\pi}_1 \hat{\pi}_2 \hat{\pi}_1' \hat{\pi}_2' \}$ and that $\hat{\tau}^{\hat{\rho}^2} \in \{ \hat{\tau} \hat{\pi}_1' \hat{\pi}_2', \hat{\tau} \hat{\pi}_1 \hat{\pi}_2 \hat{\pi}_1' \hat{\pi}_2' \}$. Replacing $\hat{\rho}$ by a suitable element of order 3 in $\langle \hat{\rho}, \hat{\mu}, \hat{\pi}_1 \rangle$, we may assume that $\hat{\rho} \in N_{\hat{C}}(\langle \hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_1' \hat{\pi}_2', \hat{\pi}_3', \hat{\mu}, \hat{\mu}', \hat{\tau} \rangle) \cap C_{\hat{C}}(\langle \hat{\mu}', \hat{\pi}_3' \rangle)$ with $\hat{\tau}^{\hat{\rho}} = \hat{\pi}_1' \hat{\pi}_2'$, $(\hat{\pi}_1' \hat{\pi}_2')^{\hat{\rho}} = \hat{\tau} \hat{\pi}_1' \hat{\pi}_2'$.

Let $\rho \in C_G(Z(S)) \cap N_G(\langle \pi_1, \pi_2, \pi_3, \pi_1' \pi_2', \pi_3', \mu, \mu', \tau \rangle)$ such that $\hat{\rho}$ is the image of ρ in \hat{C} . Then ρ acts as follows:

x	x^{ρ}
π_1	$\mu \pi_2$
π_2	$\mu \pi_1$
π_3	π_3
$\pi_1' \pi_2'$	$\tau \pi_1' \pi_2'$
π_3'	π_3'
μ	π_1
μ'	μ'
τ	$\pi_1' \pi_2'$

Comparing the actions of $(\pi_2'\pi_3')^\rho$ and $\rho\pi_2'\pi_3'$ on $\langle\pi_1, \pi_2, \pi_3, \pi_1'\pi_2', \pi_3', \mu, \mu', \tau\rangle$, we conclude that $(\pi_2'\pi_3')^\rho \equiv \rho\pi_2'\pi_3' \pmod{Z(S) \times O(C_G(Z(S)))}$. Thus $\rho^{-1}(\pi_2'\pi_3')\rho \equiv \rho\pi_2'\pi_3'z \pmod{O(C_G(Z(S)))}$ for some $z \in Z(S)$. So $\rho^{\pi_2'\pi_3'} \equiv \rho^2z \pmod{O(C_G(Z(S)))}$. Thus $z = 1$, whence

$$(\pi_2'\pi_3')^\rho \equiv \rho\pi_2'\pi_3' \pmod{O(C_G(Z(S)))}.$$

We fix ρ as above for the remainder of the paper.

LEMMA 6.4. $C_G(\alpha)/\langle\alpha\rangle \times O(C_G(\alpha))$ has a normal subgroup of index 2 isomorphic to the nontrivial split extension of an E_{16} by \mathcal{O}_6 .

$$C_G(\alpha)/O(C_G(\alpha)) \cong C_{\mathcal{O}_{12}}((12) (34) (56) (78) (9\ 10) (11\ 12)).$$

Proof. Let $\bar{C} = C_G(\alpha)/\langle\alpha\rangle \times O(C_G(\alpha))$.

By (3.3), the fusion of involutions of \bar{S} in \bar{C} is controlled by

$$\langle N_{\bar{C}}(\bar{A}), C_{\bar{C}}(\bar{\pi}_3) \rangle = \langle \bar{S}C_{\bar{C}}(\bar{A}), \bar{\xi}, \bar{\rho} \rangle;$$

hence by $\langle \bar{S}, \bar{\xi}, \bar{\rho} \rangle$. Let $\bar{E} = \langle \bar{\pi}_1, \bar{\pi}_3, \bar{\mu}, \bar{\mu}' \rangle$. Then $\langle \bar{S}, \bar{\rho}, \bar{\xi} \rangle \subseteq N_{\bar{C}}(\bar{E})$. By (5.4) and (6.2), the involution fusion pattern in \bar{S} with respect to \bar{C} is:

$$\bar{1}\bar{2}\bar{19}\bar{22} \mid \bar{3}\bar{4} \mid \bar{7}\bar{8}\bar{9}\bar{21}\bar{24} \mid \bar{10}\bar{11}\bar{25} \mid \bar{17}\bar{27}.$$

No involution of $\bar{S}_0 = \langle \bar{\pi}_2, \bar{\pi}_3, \bar{\pi}_1'\bar{\pi}_2', \bar{\pi}_2'\bar{\pi}_3' \rangle \langle \bar{\mu}, \bar{\mu}', \bar{\tau} \rangle$ is fused to an involution of $\bar{S} - \bar{S}_0$ in \bar{C} . It follows by Thompson's transfer lemma that $\exists \bar{C}_0 \trianglelefteq \bar{C}$ of index 2 with $\bar{S}_0 \in \text{Syl}_2(\bar{C}_0)$. Since $\bar{S}_0 = \langle \bar{S}_0 \cap (\bar{S}_0')^g \mid g \in \bar{C}_0 \rangle$, \bar{C}_0 is a fusion-simple group with Sylow 2-subgroups of type \mathcal{O}_8 and exactly two classes of involutions. Moreover, $C_{\bar{C}_0}(\bar{\pi}_3) = \bar{E} \langle \bar{\pi}_1'\bar{\pi}_2', \bar{\tau}, \bar{\rho}, \bar{\pi}_2'\bar{\pi}_3' \rangle \cong \mathfrak{S}_4 \cdot E_{16}$ by (6.3). As the centralizer of a central involution in $\mathcal{O}_7 \cdot E_{16}$ is isomorphic to $L_2(7) \cdot E_{16}$, \bar{C}_0 is a splitting extension of \bar{E} by \mathcal{O}_6 , by Theorem A of Gorenstein-Harada [13]. As \mathcal{O}_6 has no subgroup of index 3 or 5 and as $\bar{\xi} \notin C_{\bar{C}}(\bar{\pi}_3)$, $C_G(\alpha)/O(C_G(\alpha)) = \langle \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_3, \bar{\mu}, \bar{\mu}' \rangle \langle \bar{\pi}_1', \bar{\pi}_2', \bar{\pi}_3', \bar{\tau}, \bar{\rho}, \bar{\xi} \rangle$. The map $\theta : C_G(\alpha)/O(C_G(\alpha)) \rightarrow \mathcal{O}_{12}$ given by

$$\begin{aligned} \theta : \bar{\pi}_1 &\rightarrow (1\ 2) (3\ 4) \\ \bar{\pi}_2 &\rightarrow (5\ 6) (7\ 8) \\ \bar{\pi}_3 &\rightarrow (9\ 10) (11\ 12) \\ \bar{\pi}_1' &\rightarrow (1\ 3) (2\ 4) \\ \bar{\pi}_2' &\rightarrow (5\ 7) (6\ 8) \\ \bar{\pi}_3' &\rightarrow (9\ 11) (10\ 12) \\ \bar{\mu} &\rightarrow (1\ 2) (5\ 6) \\ \bar{\mu}' &\rightarrow (1\ 2) (9\ 10) \\ \bar{\tau} &\rightarrow (1\ 5) (2\ 6) (3\ 7) (4\ 8) \\ \bar{\rho} &\rightarrow (3\ 7\ 5) (4\ 8\ 6) \\ \bar{\xi} &\rightarrow (1\ 5\ 9) (2\ 6\ 10) (3\ 7\ 11) (4\ 8\ 12) \end{aligned}$$

defines an isomorphism of $C_G(\alpha)/O(C_G(\alpha))$ onto

$$C\alpha_{12}((12) (34) (56) (78) (9\ 10) (11\ 12))$$

by the above and Yamaki [17].

Case 6A. $\bar{N} \cong L_2(7)$.

LEMMA 6.5. $C_G(\pi_3) = O(C_G(\pi_3)) C_G(Z(S))$.

Proof. $N_{C_G(\pi_3)}(A) = SC_G(A)$. Hence, by the fusion determined in (6.2), α is weakly closed in S with respect to $C_G(\pi_3)$. The lemma follows by the Z^* -theorem.

LEMMA 6.6. $C_G(\pi_1\pi_2)$ is solvable.

Proof. By (3.3), $N_{C_G(\pi_1\pi_2)}(A)$ and $C_G(Z(S))$ control fusion of involutions in S with respect to $C_G(\pi_1\pi_2)$.

$$N_{C_G(\pi_1\pi_2)}(A)/C_G(A) \cong \mathfrak{S}_4 \quad \text{and} \quad \bar{\mu}' \notin N_{C_G(\pi_1\pi_2)}(A)/C_G(A))'.$$

So $\mu'a$ is not fused into $A\langle\mu, \tau\rangle$ in $C_G(\pi_1\pi_2)$ for all $a \in C_{A\langle\mu\rangle}(\mu')$ and, by (5.5) and (6.2), the fusion of involutions in $A\langle\mu, \tau\rangle$ with respect to $C_G(\pi_1\pi_2)$ is

$$1-10-19-25 \mid 2-4 \mid 3-11 \mid 5 \mid 6-18-20-28 \mid 7-15 \mid 8-16-17-21-26-27 \mid 9 \mid \\ 12-13 \mid 14.$$

Let $\bar{C} = C_G(\pi_1\pi_2)/\langle\pi_1\pi_2\rangle \times O(C_G(\pi_1\pi_2))$. $N_{\bar{C}}(\bar{S}) = \bar{S}C_{\bar{C}}(\bar{S})$ and $\bar{S}_0 = \langle\bar{S} \cap (\bar{S}')^g \mid g \in \bar{C}\rangle = \langle\bar{\pi}_1, \bar{\pi}_3, \bar{\pi}_1'\bar{\pi}_2', \bar{\pi}_3', \bar{\mu}, \bar{\tau}\rangle$. So by Grün's theorem, $\exists \bar{C}_0 \trianglelefteq \bar{C}$ of index 4 with $\bar{S}_0 \in \text{Syl}_2(\bar{C}_0)$. Now \bar{S}_0 is abelian, $C_{\bar{C}_0}(\bar{\pi}_3) = O(C_{\bar{C}_0}(\bar{\pi}_3))\langle\bar{S}_0, \bar{\rho}\rangle$, and $\bar{\pi}_3$ has exactly 3 conjugates in \bar{S}_0 with respect to \bar{C}_0 . Hence,

$$N_{\bar{C}_0}(\bar{S}_0) = C_{\bar{C}_0}(\bar{S}_0)\langle\bar{\rho}, \bar{\omega}\rangle.$$

Suppose that \bar{L} is a simple component of $O^{2'}(C_0)$. A 3-element of $N_{\bar{L}}(\bar{S}_0)$ must centralize a 4-dimensional subspace of \bar{S}_0 . But $C_{\bar{S}_0}(\bar{\rho}) = \langle\bar{\pi}_3, \bar{\pi}_3'\rangle$, $C_{\bar{S}_0}(\bar{\omega}) = \langle\bar{1}\rangle$ and $C_{\bar{S}_0}(\bar{\omega}\bar{\rho}) = \langle\bar{\mu}\bar{\pi}_1'\bar{\pi}_2', \bar{\tau}\bar{\mu}\bar{\pi}_1\rangle$, a contradiction. Thus $O^2(\bar{C}_0) = \bar{S}_0$, whence \bar{C} is solvable.

LEMMA 6.7. $C_G(\alpha')$ is solvable.

Proof. $N_{C_G(\alpha')}(A) = C_G(A)\langle\tau, \xi\rangle \subseteq C_G(\alpha)$. So α is isolated in S with respect to $C_G(\alpha')$. Thus by the Z^* -theorem, $C_G(\alpha') = O(C_G(\alpha')) C_G(\alpha, \alpha')$. Now, by (6.4), $C_G(\alpha, \alpha')/O(C_G(\alpha, \alpha'))$ is isomorphic to

$$C\alpha_{12}((12) (34) (56) (78) (9\ 10) (11\ 12), (13) (24) (57) (68) (9\ 11) (10\ 12)).$$

$\mathfrak{S}_{\{1,3,5,7,9,11\}}$ maps onto $C_{\mathcal{O}_{12}}(\theta(\alpha))/O_{2',2}(C_{\mathcal{O}_{12}}(\theta(\alpha)))$. $C_{\mathfrak{S}_{\{1,3,5,7,9,11\}}}(\theta(\alpha'))$ is isomorphic to $\mathfrak{S}_{\{1,5,9\}}$. Thus $C_G(\alpha, \alpha')/O_{2',2}(C_G(\alpha, \alpha')) = \langle \bar{\tau}, \bar{\xi} \rangle$. So $C_G(\alpha, \alpha')$ is solvable, whence $C_G(\alpha')$ is solvable.

THEOREM 6.8. $G \cong \text{Sp}_6(2)$.

Proof. By (6.2) and (6.4)–(6.7), $C_G(\sigma)$ is 2-constrained for every involution, σ , in G . So, by Gorenstein and Walter (quoted in [12, p. 210]), $O(C_G(\alpha)) = \langle 1 \rangle$. Then, by (6.2), (6.4) and a characterization theorem of Yamaki [18], $G \cong \text{Sp}_6(2)$.

So Theorem A is proved.

Case 6B. $\bar{N} \cong \bar{\mathfrak{S}}_4$.

LEMMA 6.9. $C_G(\pi_3)/O(C_G(\pi_3)) = (\langle \bar{\pi}_3, \bar{\pi}_3' \rangle \times \bar{K}) \langle \bar{\mu}' \rangle$, where $\bar{K} \cong \mathcal{O}_8$ or \mathcal{O}_9 and $\langle \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2', \bar{\mu}, \bar{\tau} \rangle \in \text{Syl}_2(\bar{K})$.

Proof. $N_{C_G(\pi_3)}(A) = C_G(A) \langle \mu, \mu', \tau, x_1, x_2 \rangle$. So, by (3.3) and (6.3), $\{\pi_3', \pi_3\pi_3'\}$ is a full $C_G(\pi_3)$ -class of S . So, by the Z^* -theorem, $O(C_G(\pi_3)) \langle \pi_3, \pi_3' \rangle \trianglelefteq C_G(\pi_3)$ and $C_0 = O(C_G(\pi_3)) C_G(\langle \pi_3, \pi_3' \rangle) \trianglelefteq C_G(\pi_3)$ of index 2 and $S_0 = \langle \pi_1, \pi_2, \pi_3, \pi_1', \pi_2', \pi_3', \mu, \tau \rangle \in \text{Syl}_2(C_0)$.

Let $\bar{C}_0 = C_0/O(C_G(\pi_3)) \langle \pi_3, \pi_3' \rangle$. Then, by (6.2) and the action of $\langle x_1, x_2 \rangle$ on A , \bar{C}_0 has the involution fusion pattern

$$\bar{\pi}_1 \sim \bar{\pi}_1' \sim \bar{\mu} \mid \bar{\pi}_1\bar{\pi}_2 \sim \bar{\pi}_1\bar{\pi}_2' \sim \bar{\pi}_1'\bar{\pi}_2' \sim \bar{\tau}.$$

Also S splits over $\langle \pi_3, \pi_3' \rangle$. So, by Gaschütz [5] and the fusion of involutions in $\bar{C} = C_G(\pi_3)/O(C_G(\pi_3))$, $\exists \bar{K} \trianglelefteq \bar{C}$ with $\bar{C} = (\langle \bar{\pi}_3, \bar{\pi}_3' \rangle \times \bar{K}) \langle \bar{\mu}' \rangle$ and $\bar{S}_1 = \langle \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2', \bar{\mu}, \bar{\tau} \rangle \in \text{Syl}_2(\bar{K})$. Now \bar{K} has the involution fusion pattern of \mathcal{O}_8 . So, by Theorem A of Gorenstein–Harada [12], $\bar{K} \cong \mathcal{O}_8$ or $\bar{K} \cong \mathcal{O}_9$.

LEMMA 6.10.

$$C_G(\pi_1\pi_2)/O(C_G(\pi_1\pi_2)) = (\langle \bar{\mu}, \bar{\pi}_1, \bar{\pi}_2 \rangle \langle \bar{\pi}_1'\bar{\pi}_2', \bar{\tau}, \bar{\rho} \rangle \times \bar{B}) \langle \bar{\mu}\bar{\mu}', \bar{\pi}_1' \rangle,$$

where $\bar{B} \cong \mathcal{O}_4$ or \mathcal{O}_5 with $\langle \bar{\pi}_3, \bar{\pi}_3' \rangle \in \text{Syl}_2(\bar{B})$.

Moreover $\bar{B} \cong \mathcal{O}_4$ if and only if $\bar{K} \cong \mathcal{O}_8$, \bar{K} as in (6.9).

Proof.

$$N_{C_G(\pi_1\pi_2)}(A) = C_G(A) \langle \mu, \mu', \tau, x_3 \rangle.$$

So $C_G(A) \langle \mu, \tau, x_3 \rangle \trianglelefteq N_{C_G(\pi_1\pi_2)}(A)$. Thus by (3.3) and (6.3), $\mu'a$ is not fused

into $A\langle\mu, \tau\rangle$ in $C_G(\pi_1\pi_2)$ for all $a \in C_{A\langle\mu\rangle}(\mu')$ and, by (6.2) and the action of $\langle x_3 \rangle$ on A , the fusion of involutions of $A\langle\mu, \tau\rangle$ in $C_G(\pi_1\pi_2)$ is

$$\begin{array}{l} 1-19 \mid 2-4 \mid 3 \mid 5 \mid 6-8-20-21 \mid 7 \mid 9-11 \mid 10-25 \mid 12-13 \mid 14-15 \mid \\ 16-17-18-26-27-28. \end{array}$$

As $N_{C_G(\pi_1\pi_2)}(S) = SC_G(S)$, Grün's theorem yields $C_0 \trianglelefteq C_G(\pi_1\pi_2)$ of index 4 with

$$S_0 = \langle \pi_1, \pi_2, \pi_3, \pi_1'\pi_2', \pi_3', \mu, \tau \rangle \in \text{Syl}_2(C_0).$$

Let $\bar{C} = C_G(\pi_1\pi_2)/\langle \pi_1\pi_2 \rangle \times O(C_G(\pi_1\pi_2))$. Then \bar{S}_0 is abelian; $C_{\bar{C}_0}(\bar{\pi}_3) = O(C_{\bar{C}_0}(\bar{\pi}_3))\bar{S}_0\langle\bar{\rho}\rangle$; and $\bar{\pi}_3$ has exactly 3 conjugates in $N_{\bar{C}_0}(\bar{S}_0)$; whence $N_{\bar{C}_0}(\bar{S}_0) = \bar{S}_0\langle\bar{\rho}, \bar{x}_3\rangle$. Now \bar{x}_3 centralizes $\langle\bar{\pi}_1, \bar{\pi}_1'\bar{\pi}_2', \bar{\mu}, \bar{\rho}\rangle$ and permutes $\langle\bar{\pi}_3, \bar{\pi}_3'\rangle^\#$. Also $\bar{\rho}$ centralizes $\langle\bar{\pi}_3, \bar{\pi}_3'\rangle$ and permutes $\langle\bar{\pi}_1, \bar{\pi}_1'\bar{\pi}_2', \bar{\mu}, \bar{\tau}\rangle^\#$ fixed-point freely. Hence either $O^{2'}(\bar{C}_0) = \bar{S}_0$ or

$$O^{2'}(\bar{C}_0) = \bar{B} \times \langle \pi_1, \pi_1'\pi_2', \bar{\mu}, \bar{\tau} \rangle$$

with \bar{B} simple. As $\bar{\pi}_3$ is the only element of the coset $\langle\bar{\pi}_1, \bar{\pi}_1'\bar{\pi}_2', \bar{\mu}, \bar{\tau}\rangle\bar{\pi}_3$ having only 3 conjugates in $N_{\bar{C}_0}(\bar{S}_0)$, $\langle\bar{\pi}_3, \bar{\pi}_3'\rangle \in \text{Syl}_2(\bar{B})$. Now $C_G(\pi_1)/O(C_G(\pi_1)) = \langle\bar{\pi}_1, \bar{\pi}_1'\rangle \times \bar{L}\langle\bar{\mu}\rangle$, where $\bar{L} \cong \mathcal{O}_8$ or $\bar{L} \cong \mathcal{O}_9$ and $\langle\bar{\pi}_2, \bar{\pi}_3, \bar{\pi}_2', \bar{\pi}_3', \bar{\mu}\bar{\mu}', \bar{\tau}^{\bar{\epsilon}}\rangle \in \text{Syl}_2(\bar{L})$. If $\bar{L} \cong \mathcal{O}_8$, $C_{\bar{L}}(\langle\bar{\pi}_2, \bar{\pi}_2'\rangle)$ is solvable. So $O^{2'}(\bar{C}_0) = \bar{S}_0 \trianglelefteq \bar{C}_0$ of index 9 and we are done in this case.

Now suppose that $\bar{L} \cong \mathcal{O}_9$.

$$C_G(\langle\pi_1, \pi_2, \mu\rangle)/O(C_G(\pi_1)) \cap C_G(\langle\pi_1, \pi_2, \mu\rangle) \cong C_G(\pi_1)/O(C_G(\pi_1)).$$

So

$$C_G(\langle\pi_1, \pi_2, \mu\rangle)/O(C_G(\pi_1)) \cap C_G(\langle\pi_1, \pi_2, \mu\rangle) \cong Z_2 \times Z_2 \times Z_2 \times \mathfrak{S}_5.$$

Since

$$\begin{aligned} O(C_G(\langle\pi_1, \pi_2, \mu\rangle)) &\supseteq O(C_G(\pi_1)) \cap C_G(\langle\pi_1, \pi_2, \mu\rangle), \\ C_G(\langle\pi_1, \pi_2, \mu\rangle)/O(C_G(\pi_1)) &\cong Z_2 \times Z_2 \times Z_2 \times \mathfrak{S}_5. \end{aligned}$$

Now $C_G(\langle\pi_1, \pi_2, \mu\rangle) \subseteq C_G(\pi_1\pi_2)$. As \bar{B}_0 is the only not-necessarily-solvable composition factor of $C_G(\pi_1\pi_2)$, $\bar{B}_0 \cong \mathcal{O}_5$ by the Jordan-Hölder theorem. Thus $O^{2'}(\bar{C}_0) = \langle\bar{\pi}_1, \bar{\pi}_1'\bar{\pi}_2', \bar{\mu}, \bar{\tau}\rangle \times \bar{B}_0 \trianglelefteq \bar{C}_0$ of index 3 with $\bar{B}_0 \cong \mathcal{O}_5$ and $\langle\bar{\pi}_3, \bar{\pi}_3'\rangle \in \text{Syl}_2(\bar{B}_0)$. Hence the conclusion holds in both cases.

COROLLARY 6.11. *O is an A-signalizer functor.*

Proof. As $\langle\bar{\pi}_3, \bar{\pi}_3'\rangle \in \text{Syl}_2(\bar{B})$ with \bar{B} as in (6.10),

$$\text{H}_{C_G(\pi_1\pi_2)/O(C_G(\pi_1\pi_2))}(\bar{A}, 2') = \langle\bar{I}\rangle.$$

If $A^\# \ni a \sim_G \pi_1\pi_2$, then $a \sim_{N_G(A)} \pi_1\pi_2$. So $\text{H}_{C_G(a)/O(C_G(a))}(\bar{A}, 2') = \langle\bar{I}\rangle$ for all $a \in A^\#$, $a \sim_G \pi_1\pi_2$.

As $C_G(a)$ is 2-constrained for all $a \in A^\#$, $a \sim_G \alpha$, it remains to prove that $O(C_G(a)) \cap C_G(b) \subseteq O(C_G(b))$ for all $a \in A^\#$, $b \in A^\#$, $b \sim_G \pi_3$.

Let $\bar{C} = C_G(\pi_3)/O(C_G(\pi_3)) = \langle \bar{\pi}_3, \bar{\pi}_3' \rangle \times \bar{K}$ with \bar{K} as in (6.9). Then $C_{\bar{C}}(\bar{\pi}_3') = \langle \bar{\pi}_3, \bar{\pi}_3' \rangle \times \bar{K}$. So $O(C_{\bar{C}}(\bar{\pi}_3')) = \langle \bar{1} \rangle$.

Let $\bar{C}_0 = \langle \bar{\pi}_3, \bar{\pi}_3' \rangle \times \bar{K}$.

For

$$\bar{a} \in \langle \bar{\pi}_1, \bar{\pi}_2, \bar{\pi}_1', \bar{\pi}_2' \rangle,$$

$$C_{\bar{C}_0}(\bar{a}) = C_{\bar{C}_0}(\bar{a}\bar{\pi}_3) = C_{\bar{C}_0}(\bar{a}\bar{\pi}_3') = C_{\bar{C}_0}(\bar{a}\bar{\pi}_3\bar{\pi}_3') = \langle \bar{\pi}_3, \bar{\pi}_3' \rangle \times C_{\bar{K}}(\bar{a}).$$

Now $O(C_{\bar{K}}(\bar{a})) = \langle \bar{1} \rangle$ for $\bar{K} \cong \mathcal{O}_8$ or \mathcal{O}_9 and \bar{a} any involution. So $O(C_{\bar{C}_0}(\bar{a})) = \langle \bar{1} \rangle$. It follows that $O(C_{\bar{C}}(\bar{a})) = \langle \bar{1} \rangle$ for all $\bar{a} \in \bar{A}^\#$.

Now let $b \in A^\#$, $b \sim_G \pi_3$ and let $n \in N_G(A)$ such that $\pi_3^n = b$. Then $C_G(b)/O(C_G(b)) = C_G(\pi_3^n)/O(C_G(\pi_3))^n = C_0^n \langle \bar{\mu}'^n \rangle$ with $C_0^n = \langle b, \bar{\pi}_3'^n \rangle \times \bar{K}^n$, $\bar{\pi}_3'^n \in \bar{A}$ and $\bar{A} \subseteq C_0^n$. So the same argument proves that

$$O(C_G(a)) \cap C_G(b) \subseteq O(C_G(b))$$

for all $a, b \in A^\#$, $b \sim_G \pi_3$.

Thus O is an A -signalizer functor.

Let $W_A = \langle O(C_G(a)) \mid a \in A^\# \rangle$. By Goldschmidt [8], W_A has odd order.

Now assume that G is a minimal counterexample to Theorem B.

THEOREM 6.12. *No minimal counterexample exists.*

Proof. Suppose that G is such. If $W_A = \langle 1 \rangle$, then $O(C_G(\alpha)) = \langle 1 \rangle$. But then by (6.2), (6.4) and a characterization theorem of Yamaki [17], $G \cong \mathcal{O}_{12}$ or \mathcal{O}_{13} . So $W_A \neq \langle 1 \rangle$; hence $N_G(W_A) < G$. By Lemma 5.5 of Gorenstein–Harada [12], $N_G(A) \subseteq N_G(W_A)$ and $C_G(Z(S)) \subseteq N_G(W_A)$. So, by (3.3), $N_G(W_A)$ has the involution fusion pattern of \mathcal{O}_{12} . By the minimality of G as a counterexample to Theorem B, $N_G(W_A)/O(N_G(W_A)) \cong \mathcal{O}_{12}$ or \mathcal{O}_{13} .

Now $C_G(\alpha)/O(C_G(\alpha)) \cong C_{\mathcal{O}_{13}}((12) (34) (56) (78) (9 \ 10) (11 \ 12))$. So $|C_G(\alpha)| = |C_{N_G(W_A)}(\alpha)|$; hence $C_G(\alpha) \subseteq N_G(W_A)$. Moreover, if $N_G(W_A)/O(N_G(W_A)) \cong \mathcal{O}_{13}$, then by comparison of orders, $N_G(W_A)$ contains $C_G(\sigma)$ for all $\sigma \in Z(S)^\#$; hence, by the involution fusion pattern in $N_G(W_A)$, for all involutions in $N_G(W_A)$. So $N_G(W_A)$ is strongly embedded in G . But G has three conjugacy classes of involutions, a contradiction.

So $N_G(W_A)/O(N_G(W_A)) \cong \mathcal{O}_{12}$.

But $C_G(\pi_3) \subseteq O(C_G(\pi_3))$, $N_G(\langle \pi_3, \pi_3' \rangle) \subseteq N_G(W_A)$ by (6.9) and Lemma 5.5 of Gorenstein–Harada [12]. Then necessarily, $\bar{K} \cong \mathcal{O}_8$. So $\bar{B} \cong \mathcal{O}_4$, whence by (6.10),

$$C_G(\pi_1\pi_2) \subseteq O(C_G(\pi_1\pi_2)) N_G(\langle \pi_3, \pi_3' \rangle) \subseteq N_G(W_A).$$

But then $N_G(W_A)$ is strongly embedded in G , a contradiction.

This proves Theorem B.

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